

Noise recovery for Lévy-driven CARMA processes and high-frequency behaviour of approximating Riemann sums

Vincenzo Ferrazzano ^{*} Florian Fuchs[†]

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Abstract

We consider high-frequency sampled continuous-time autoregressive moving average (CARMA) models driven by finite-variance zero-mean Lévy processes. An L^2 -consistent estimator for the increments of the driving Lévy process without order selection in advance is proposed if the CARMA model is invertible. In the second part we analyse the high-frequency behaviour of approximating Riemann sum processes, which represent a natural way to simulate continuous-time moving average processes on a discrete grid. We shall compare their autocovariance structure with the one of sampled CARMA processes, where the rule of integration plays a crucial role. Moreover, new insight into the kernel estimation procedure of Brockwell et al. (2012a) is given.

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1 Introduction

The constantly increasing availability of high-frequency data in finance and sciences in general has sparked in the last decade a great deal of attention about the asymptotic behaviour of high-frequency sampled processes, especially concerning the estimation of the multi-power variations of Itô semimartingales (see, e.g., Andersen and Todorov (2010), Barndorff-Nielsen and Shephard (2003)), employing their realized counterparts. These quantities are of primary importance to practitioners, since they embody the deviation of data from a Brownian motion. Such methods are summarized in the book of Jacod and Protter (2012) and references therein, which represents the most recent review on the subject.

^{*}Center for Mathematical Sciences, Technische Universität München, 85748 Garching b. München, Germany, email: ferrazzano@ma.tum.de, <http://www-m4.ma.tum.de>

[†]Institute of Mathematical Finance, Ulm University, 89081 Ulm and International School of Applied Mathematics, Technische Universität München, 85748 Garching b. München, Germany, email: ffuchs@ma.tum.de, <http://www-m4.ma.tum.de>

In many areas of application Lévy-driven processes are used for modelling time series. An ample class within this group are continuous-time moving average (CMA) processes

$$Y_t = \int_{-\infty}^{\infty} g(t-s) dL_s, \quad t \in \mathbb{R}, \quad (1.1)$$

where g is a so-called kernel function and $L = \{L_t\}_{t \in \mathbb{R}}$ is said to be the driving Lévy process (see, e.g., Sato (1999) for a detailed introduction). They cover, for instance, Ornstein-Uhlenbeck and continuous-time autoregressive moving average (CARMA) processes. The latter are the continuous-time analog of the well-known ARMA processes (see, e.g., Brockwell and Davis (1991)) and have been extensively studied over the recent years (cf. Brockwell (2001, 2004); Brockwell and Lindner (2009); Todorov and Tauchen (2006)). Originally, driving processes of CARMA models were restricted to Brownian motion (see Doob (1944), and also Doob (1990)); however, Brockwell (2001) allowed for Lévy processes with a finite r th moment for some $r > 0$.

Lévy-driven CARMA processes are widely used in various areas of application like signal processing and control (cf. Garnier and Wang (2008); Larsson et al. (2006)), high-frequency financial econometrics (cf. Todorov (2009)) and financial mathematics (cf. Benth et al. (2010); Brockwell et al. (2006); Haug and Czado (2007); Todorov and Tauchen (2006)). Stable CARMA processes can be relevant in modelling energy markets (cf. Benth et al. (2011); García et al. (2011)). Very often, a correct specification of the driving Lévy process is of primary importance in all these applications.

In this paper we will be concerned with a high-frequency sampled CARMA process driven by a second-order zero-mean Lévy process. Under the assumption of *invertibility* of the CARMA model, we shall present an L^2 -consistent estimator for the increments of the driving Lévy process, employing standard time-series techniques. It is remarkable that the proposed procedure works for arbitrary autoregressive and moving average orders, i.e. there is no need for *order selection* in advance. In the light of the results in Brockwell et al. (2012a) and the flexibility of CARMA processes, the method might apply to a wider class of CMA processes, too. Moreover, since the proof employs only the fact that the increments of the Lévy process are orthogonal rather than independent, the result holds for a much ampler class of driving processes. Notable examples are the COGARCH processes (Brockwell et al. (2006); Klüppelberg et al. (2004)) or time-changed Lévy processes (Carr et al. (2003)), which are often employed to model volatility clustering in finance and intermittency in turbulence.

This noise recovery result will give rise to the conjecture that the sampled CARMA process behaves on a high-frequency time grid approximately like a suitable $\text{MA}(\infty)$ process, which we shall call *approximating Riemann sum process*. By comparing the asymptotic properties of the autocovariance structure of high-frequency sampled CARMA models with the one of their approximation Riemann sum processes, it will turn out that the so-called *rule* of the Riemann sums plays a crucial role if the difference between the autoregressive and moving average order is greater than 1. On the one hand, this gives new insight for the kernel estimation procedure studied in Brockwell et al. (2012a) and explains at which points the kernel is indeed estimated. On the other hand, this has obvious consequences for simulation purposes. Riemann sum approximations are an easy tool to simulate CMA processes; however, our results show that one has to be careful with the chosen rule of integration in the context of certain CARMA processes.

The outline of the paper is as follows. In Chapter 2 we are going to recall the definition of finite-variance CARMA models and summarise important properties of high-frequency sampled CARMA processes. In particular, we fix a global assumption that guarantees causality and

invertibility for the sampled sequence. In the third chapter we then derive an L^2 -consistent estimator for the increments of the driving Lévy process starting from the Wold representation of the sampled process. It will turn out that *invertibility* of the original continuous-time process is sufficient and necessary for the recovery result to hold. Chapter 3 is completed by an illustrating example for CAR(2) and CARMA(2, 1) processes. Thereafter, the high-frequency behaviour of approximating Riemann sum processes is studied in Chapter 4. First, an ARMA representation for the Riemann sum approximation is established in general and then the role of the rule of integration is analysed by matching the asymptotic autocovariance structure of sampled CARMA processes and their Riemann sum approximations in the cases where the autoregressive order is less or equal to 3. The connection between the Wold representation and the approximating Riemann sum yields a deeper insight into the kernel estimation procedure introduced in Brockwell et al. (2012a). A short conclusion can be found in Chapter 5 and some auxiliary results are established in the appendix.

2 Preliminaries

2.1 Finite-variance CARMA processes

Throughout this paper we shall be concerned with a CARMA process driven by a second-order zero-mean Lévy process $L = \{L_t\}_{t \in \mathbb{R}}$ with $EL_1 = 0$ and $EL_1^2 = 1$. It is defined as follows.

For non-negative integers p and q such that $q < p$, a CARMA(p, q) process $Y = \{Y_t\}_{t \in \mathbb{R}}$ with coefficients $a_1, \dots, a_p, b_0, \dots, b_q \in \mathbb{R}$ and driving Lévy process L is defined to be a strictly stationary solution of the suitably interpreted formal equation

$$a(D)Y_t = \sigma b(D)DL_t, \quad t \in \mathbb{R}, \quad (2.1)$$

where D denotes differentiation with respect to t , $a(\cdot)$ and $b(\cdot)$ are the characteristic polynomials,

$$a(z) := z^p + a_1 z^{p-1} + \dots + a_p \quad \text{and} \quad b(z) := b_0 + b_1 z + \dots + b_{p-1} z^{p-1},$$

the coefficients b_j satisfy $b_q = 1$ and $b_j = 0$ for $q < j < p$, and $\sigma > 0$ is a constant. The polynomials $a(\cdot)$ and $b(\cdot)$ are assumed to have no common zeroes. Throughout this paper we shall denote by λ_i , $i = 1, \dots, p$, and $-\mu_i$, $i = 1, \dots, q$, the roots of $a(\cdot)$ and $b(\cdot)$ respectively, such that these polynomials can be written as $a(z) = \prod_{i=1}^p (z - \lambda_i)$ and $b(z) = \prod_{i=1}^q (z + \mu_i)$. Throughout this paper we will further assume

Assumption 1. (i) *The zeroes of the polynomial $a(\cdot)$ satisfy $\Re(\lambda_j) < 0$ for every $j = 1, \dots, p$,*
(ii) *the roots of $b(\cdot)$ have non-vanishing real part, i.e. $\Re(\mu_j) \neq 0$ for all $j = 1, \dots, q$.*

Since the derivative DL_t does not exist in the usual sense, we interpret (2.1) as being equivalent to the observation and state equations

$$Y_t = \mathbf{b}^T \mathbf{X}_t, \quad (2.2)$$

$$d\mathbf{X}_t = A\mathbf{X}_t dt + \mathbf{e}_p dL_t, \quad (2.3)$$

where

$$\mathbf{X}_t = \begin{pmatrix} X(t) \\ X^{(1)}(t) \\ \vdots \\ X^{(p-2)}(t) \\ X^{(p-1)}(t) \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{pmatrix}, \quad \mathbf{e}_p = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \dots & -a_1 \end{pmatrix} \quad \text{and } A = -a_1 \text{ for } p = 1.$$

It is easy to check that the eigenvalues of the matrix A are the same as the zeroes of the autoregressive polynomial $a(\cdot)$.

Under Assumption 1 (i) it has been shown in (Brockwell and Lindner (2009), Theorem 3.3) that Eqs. (2.2)-(2.3) have the unique strictly stationary solution,

$$Y_t = \int_{-\infty}^{\infty} g(t-u) dL_u, \quad t \in \mathbb{R}, \quad (2.4)$$

where

$$g(t) = \begin{cases} \frac{\sigma}{2\pi i} \int_{\rho} \frac{b(z)}{a(z)} e^{tz} dz = \sigma \sum_{\lambda} \text{Res}_{z=\lambda} \left(e^{zt} \frac{b(z)}{a(z)} \right), & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases} \quad (2.5)$$

and ρ is any simple closed curve in the open left half of the complex plane encircling the zeroes of $a(\cdot)$. The sum is over the distinct zeroes λ of $a(\cdot)$ and $\text{Res}_{z=\lambda}(\cdot)$ denotes the residue at λ of the function in brackets. The kernel g can also be expressed (Brockwell and Lindner (2009), equations (2.10) and (3.7)) as

$$g(t) = \sigma \mathbf{b}^\top e^{At} \mathbf{e}_p \mathbf{1}_{(0,\infty)}(t), \quad t \in \mathbb{R}, \quad (2.6)$$

and its Fourier transform is

$$\mathcal{F}\{g(\cdot)\}(\omega) := \int_{\mathbb{R}} g(s) e^{i\omega s} ds = \sigma \frac{b(-i\omega)}{a(-i\omega)}, \quad \omega \in \mathbb{R}. \quad (2.7)$$

In the light of Eqs. (2.4)-(2.7), we can interpret a CARMA process as a continuous-time filtered white noise, whose transfer function has a finite number of poles and zeros. We outline that the request on the roots of $a(\cdot)$ to lie in the interior of the left half of the complex plane in order to have causality arises from Theorem V, p. 8, Paley and Wiener (1934), which is intrinsically connected with the Theorems in Titchmarsh (1948), pp. 125-129, on the Hilbert transform. A similar request on the roots of $b(\cdot)$ will turn out to be necessary for recovering the driving Lévy process.

2.2 Properties of high-frequency sampled CARMA processes

We now recall some properties of the sampled process $Y^\Delta := \{Y_{n\Delta}\}_{n \in \mathbb{Z}}$ of a CARMA(p, q) process where $\Delta > 0$; cf. Brockwell et al. (2012a,b) and references therein. It is known that the

sampled process Y^Δ satisfies the ARMA($p, p-1$) equations

$$\Phi_\Delta(B)Y_n^\Delta = \Theta_\Delta(B)Z_n^\Delta, \quad n \in \mathbb{Z}, \quad \{Z_n^\Delta\} \sim \text{WN}(0, \sigma_\Delta^2), \quad (2.8)$$

with the AR part $\Phi_\Delta(B) := \prod_{i=1}^p (1 - e^{\Delta\lambda_i} B)$, where B is the discrete-time backshift operator, $BY_n^\Delta := Y_{n-1}^\Delta$. Finally, the MA part $\Theta_\Delta(\cdot)$ is a polynomial of order $p-1$, chosen in such a way that it has no roots inside the unit circle. For $p > 3$ and fixed $\Delta > 0$ there is no explicit expression for the coefficients of $\Theta_\Delta(\cdot)$ nor the white noise process Z^Δ . Nonetheless, asymptotic expressions for $\Theta_\Delta(\cdot)$ and $\sigma_\Delta^2 = \text{var}(Z_n^\Delta)$ as $\Delta \downarrow 0$ were obtained in Brockwell et al. (2012a,b). Namely we have that the polynomial $\Theta_\Delta(z)$ can be written as (see Theorem 2.1, Brockwell et al. (2012a))

$$\Theta_\Delta(z) = \prod_{i=1}^{p-q-1} (1 + \eta(\xi_i)z) \prod_{k=1}^q (1 - \zeta_k z), \quad z \in \mathbb{C}, \quad (2.9)$$

$$\sigma_\Delta^2 = \frac{\sigma^2 \Delta^{2(p-q)-1}}{(2(p-q)-1)! \prod_{i=1}^{p-q-1} \eta(\xi_i)} (1 + o(1)), \quad \text{as } \Delta \downarrow 0, \quad (2.10)$$

where, again as $\Delta \downarrow 0$,

$$\begin{aligned} \zeta_k &= 1 \pm \mu_k \Delta + o(\Delta), & k &= 1, \dots, q, \\ \eta(\xi_i) &= \xi_i - 1 \pm \sqrt{(\xi_i - 1)^2 - 1} + o(1), & i &= 1, \dots, p-q-1. \end{aligned} \quad (2.11)$$

The signs \pm in (2.11) are chosen respectively such that, for sufficiently small Δ , the coefficients ζ_k and $\eta(\xi_i)$ are less than 1 in absolute value, to ensure that Eq. (2.8) is invertible. Moreover, ξ_i , $i = 1, \dots, p-q-1$, are the zeros of $\alpha_{p-q-1}(\cdot)$, which is defined as the $(p-q-1)$ -th coefficient in the series expansion

$$\frac{\sinh(z)}{\cosh(z) - 1 + x} = \sum_{k=0}^{\infty} \alpha_k(x) z^{2k+1}, \quad z \in \mathbb{C}, \quad x \in \mathbb{R} \setminus \{0\}, \quad (2.12)$$

where the L.H.S. of Eq. (2.12) is a power transfer function arising from the sampling procedure (cf. Brockwell et al. (2012b), Eq. (11)). Therefore the coefficients $\eta(\xi_i)$, $i = 1, \dots, p-q-1$, can be regarded as spurious, as they do not depend on the parameters of the underlying continuous-time process Y , but just on $p-q$.

Remark 2.1. *Our notion of sampled process is a weak one, since we only require that the sampled process has the same autocovariance structure as the continuous-time process, when observed on a discrete grid. We know that the filtered process on the LHS of (2.8) (Brockwell and Lindner (2009), Lemma 2.1) is a $(p-1)$ -dependent discrete-time process. Therefore there exist 2^{p-1} possible representations for the RHS of (2.8) yielding the same autocovariance function of the filtered process, but only one having its roots outside the unit circle, which is called minimum-phase spectral factor (see Sayed and Kailath (2001) for a review on the topic). Since it is not possible to discriminate between them, we always take the minimum-phase spectral factor without any further question. This will be crucial in our main result.*

Moreover, the rationale behind Assumption 1 (ii) becomes clear now: if $\Re(\mu_k) = 0$ for some k , then the corresponding $|\zeta_k|^2$ is equal to $1 + \Delta^2 |\mu_k|^2 + o(\Delta^2)$, for either sign choice. Then the MA($p-1$) polynomial in Eq. (2.9) will never be invertible for small Δ .

In order to ensure that the sampled CARMA process is invertible, we need to verify that $|\eta(\xi_i)|$ will be strictly less than 1 for all $i = 1, \dots, p - q - 1$ and sufficiently small Δ .

Proposition 2.2. *The coefficients $\eta(\xi_i)$ in Eq. (2.11) are uniquely determined by*

$$\eta(\xi_i) = \xi_i - 1 - \sqrt{(\xi_i - 1)^2 - 1} + o(1), \quad i = 1, \dots, p - q - 1,$$

and we have that $\xi_i - 1 - \sqrt{(\xi_i - 1)^2 - 1} \in (0, 1)$ for all $i = 1, \dots, p - q - 1$.

Proof. It follows from Proposition A.1 that $\xi_i \in (2, \infty)$ for all $i = 1, \dots, p - q - 1$. This yields $\xi_i - 1 + \sqrt{(\xi_i - 1)^2 - 1} > 1$ for all i and hence, we have that

$$\eta(\xi_i) = \xi_i - 1 - \sqrt{(\xi_i - 1)^2 - 1} + o(1), \quad i = 1, \dots, p - q - 1.$$

Since the first-order term of $\eta(\xi_i)$ is positive and monotonously decreasing in ξ_i , the additional claim follows. \square

3 Noise recovery

In this section we are going to prove the first main statement of the paper, which is a recovery result for the driving Lévy process. We start with some motivation for our approach.

We know that the sampled CARMA process $Y^\Delta = \{Y_{n\Delta}\}_{n \in \mathbb{Z}}$ has the Wold representation (cf. Brockwell and Davis (1991), p. 187)

$$Y_n^\Delta = \sum_{j=0}^{\infty} \psi_j^\Delta Z_{n-j}^\Delta = \sum_{j=0}^{\infty} \left(\frac{\sigma_\Delta}{\sqrt{\Delta}} \psi_j^\Delta \right) \left(\frac{\sqrt{\Delta}}{\sigma_\Delta} Z_{n-j}^\Delta \right), \quad n \in \mathbb{Z}, \quad (3.1)$$

where $\sum_{j=0}^{\infty} (\psi_j^\Delta)^2 < \infty$. Moreover, Eq. (3.1) is the causal representation of Eq. (2.8), and it has been shown in Brockwell et al. (2012a) that for every causal and invertible CARMA(p, q) process, as $\Delta \downarrow 0$,

$$\frac{\sigma_\Delta}{\sqrt{\Delta}} \psi_{[t/\Delta]}^\Delta \rightarrow g(t), \quad t \geq 0, \quad (3.2)$$

where g is the kernel in the moving average representation (2.4). Given the availability of classical time-series methods to estimate $\{\psi_j^\Delta\}_{j \in \mathbb{N}}$ and σ_Δ^2 , and the flexibility of CARMA processes, we argue that this result can be applied to more general continuous-time moving average processes.

Given Eqs. (3.1) and (3.2) it is natural to investigate, whether the quantity

$$\bar{L}_n^\Delta := \frac{\sqrt{\Delta}}{\sigma_\Delta} Z_n^\Delta, \quad n \in \mathbb{Z},$$

approximates the increments of the driving Lévy process in the sense that for every fixed $t \in (0, \infty)$,

$$\sum_{i=1}^{[t/\Delta]} \bar{L}_i^\Delta \xrightarrow{L^2} L_t, \quad \Delta \downarrow 0. \quad (3.3)$$

As usual, convergence in L^2 implies also convergence in probability and distribution.

The first results on retrieving the increments of L were given in Brockwell et al. (2011), and furthermore generalized to the multivariate case by Brockwell and Schlemm (2011). The essential limitation of this parametric method is that it might not be robust with respect to

model misspecification. More precisely, the fact that a $\text{CARMA}(p, q)$ process is $(p - q - 1)$ -times differentiable (see Proposition 3.32 of Marquardt and Stelzer (2007)) is essential for the procedure to work (cf. Theorem 4.3 of Brockwell and Schlemm (2011)). However, if the underlying process is instead $\text{CARMA}(p', q')$ with $p' - q' < p - q$, then some of the necessary derivatives do not longer exist. In contrast to the aforementioned procedure, in the method we propose there is no need to specify the autoregressive and the moving average orders p and q in advance.

Before we start to prove the recovery result in Eq. (3.3), let us, in analogy to the discrete-time case, establish the notion of invertibility.

Definition 3.1. A $\text{CARMA}(p, q)$ process is said to be invertible if the roots of the moving average polynomial $b(\cdot)$ have negative real parts, i.e. $\Re(\mu_i) > 0$ for all $i = 1, \dots, q$.

Our main theorem is the following:

Theorem 3.2. Let Y be a finite-variance $\text{CARMA}(p, q)$ process and Z^Δ the noise on the RHS of the sampled Eq. (2.8). Moreover, let Assumption 1 hold and define $\bar{L}^\Delta := \sqrt{\Delta}/\sigma_\Delta Z^\Delta$. Then, as $\Delta \downarrow 0$,

$$\sum_{i=1}^{\lfloor t/\Delta \rfloor} \bar{L}_i^\Delta \xrightarrow{L^2} L_t, \quad t \in (0, \infty), \quad (3.4)$$

if and only if the roots of the moving average polynomial $b(\cdot)$ on the RHS of the CARMA Eq. (2.1) have negative real parts, i.e. if and only if the CARMA process is invertible.

Proof. Under Assumption 1 (ii) and using Proposition 2.2, the sampled ARMA equation (2.8) is invertible. The noise on the RHS of Eq. (2.8) is then obtained using the classical inversion formula

$$Z_n^\Delta = \frac{\Phi_\Delta(B)}{\Theta_\Delta(B)} Y_n^\Delta, \quad n \in \mathbb{Z},$$

where B is the usual backshift operator. Let us consider the stationary *continuous-time* process

$$\mathcal{Z}_t^\Delta := \frac{\Phi_\Delta(B_\Delta)}{\Theta_\Delta(B_\Delta)} Y_t = \sum_{i=0}^{\infty} a_i^\Delta \int_{-\infty}^{t-i\Delta} g(t-i\Delta-s) dL_s, \quad t \in \mathbb{R}, \quad (3.5)$$

where the continuous-time backshift operator B_Δ is defined such that $B_\Delta Y_t := Y_{t-\Delta}$ for every $t \in \mathbb{R}$ and the series in the RHS of Eq. (3.5) is the Laurent expansion of the rational function $\Phi_\Delta(\cdot)\Theta_\Delta^{-1}(\cdot)$. Moreover, $\mathcal{Z}_{n\Delta}^\Delta = Z_n^\Delta$ for every $n \in \mathbb{N}$; as a consequence, the random variables $\mathcal{Z}_s^\Delta, \mathcal{Z}_t^\Delta$ are uncorrelated for $|t-s| \geq \Delta$ and $\text{var}(\mathcal{Z}_t^\Delta) = \text{var}(Z_n^\Delta)$. Exchanging the sum and the integral signs in Eq. (3.5), and the fact that $g(\cdot) = 0$ for negative arguments, we have that \mathcal{Z}^Δ is a continuous-time moving average process

$$\mathcal{Z}_t^\Delta = \int_{-\infty}^t g^\Delta(t-s) dL_s, \quad t \in \mathbb{R},$$

whose kernel function g^Δ has Fourier transform (cf. Eq. (2.7))

$$\mathcal{F}\{g^\Delta(\cdot)\}(\omega) = \frac{\Phi_\Delta(e^{i\omega\Delta})}{\Theta_\Delta(e^{i\omega\Delta})} \mathcal{F}\{g(\cdot)\}(\omega) = \sigma \frac{\Phi_\Delta(e^{i\omega\Delta})}{\Theta_\Delta(e^{i\omega\Delta})} \frac{b(-i\omega)}{a(-i\omega)}, \quad \omega \in \mathbb{R}, \quad \Delta > 0.$$

Since we can write $L_t - L_{t-\Delta} = \int_{-\infty}^t \mathbf{1}_{(0,\Delta)}(t-s) dL_s$, $t \in \mathbb{R}$, the sum of the differences between the rescaled sampled noise terms and the increments of the Lévy process is given by

$$\sum_{j=1}^n \bar{L}_j^\Delta - L_{n\Delta} = \int_{-\infty}^{n\Delta} \sum_{j=1}^n \left[\frac{\sqrt{\Delta}}{\sigma_\Delta} g^\Delta(j\Delta - s) - \mathbf{1}_{(0,\Delta)}(j\Delta - s) \right] dL_s = \int_{-\infty}^{n\Delta} h_n^\Delta(n\Delta - s) dL_s, \quad (3.6)$$

where, for every $n \in \mathbb{N}$,

$$h_n^\Delta(s) := \sum_{j=1}^n \left[\frac{\sqrt{\Delta}}{\sigma_\Delta} g^\Delta(s + (j-n)\Delta) - \mathbf{1}_{(0,\Delta)}(s + (j-n)\Delta) \right], \quad s \in \mathbb{R},$$

and the stochastic integral in Eq. (3.6) w.r.t. L is still in the L^2 -sense. It is a standard result, cf. (Gikhman and Skorokhod, 2004, Ch. IV, §4), that the variance of the moving average process in Eq. (3.6) is given by

$$\mathbb{E} \left[\sum_{j=1}^n \bar{L}_j^\Delta - L_{n\Delta} \right]^2 = \int_{-\infty}^{n\Delta} (h_n^\Delta(n\Delta - s))^2 ds = \|h_n^\Delta(\cdot)\|_{L^2}^2, \quad (3.7)$$

where the latter equality is true since $h_n^\Delta(s) = 0$ for any $s \leq 0$.

Furthermore, the Fourier transform of $h_n^\Delta(\cdot)$ can be readily calculated, invoking the linearity and the shift property of the Fourier transform. We thus obtain

$$\begin{aligned} \mathcal{F}\{h_n^\Delta(\cdot)\}(\omega) &= \left[\frac{\sqrt{\Delta}}{\sigma_\Delta} \mathcal{F}\{g^\Delta(\cdot)\}(\omega) - \mathcal{F}\{\mathbf{1}_{(0,\Delta)}(\cdot)\}(\omega) \right] \sum_{j=1}^n e^{i\omega(n-j)\Delta} \\ &= \left[\sigma \frac{\sqrt{\Delta}}{\sigma_\Delta} \frac{\prod_{j=1}^p (1 - e^{\Delta(\lambda_j + i\omega)})}{\Theta_\Delta(e^{i\omega\Delta})} \frac{b(-i\omega)}{a(-i\omega)} - \frac{e^{i\omega\Delta} - 1}{i\omega} \right] \frac{1 - e^{i\omega\Delta n}}{1 - e^{i\omega\Delta}} \\ &=: [h^{\Delta,1}(\omega) - h^{\Delta,2}(\omega)] \cdot h_n^{\Delta,3}(\omega), \quad \omega \in \mathbb{R}. \end{aligned}$$

Due to Plancherel's Theorem, we deduce

$$\begin{aligned} \text{var} \left[\sum_{i=1}^n \bar{L}_i^\Delta - L_{n\Delta} \right] &= \|h_n^\Delta(\cdot)\|_{L^2}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |\mathcal{F}\{h_n^\Delta(\cdot)\}|^2(\omega) d\omega, \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left[|h^{\Delta,1} \cdot h_n^{\Delta,3}(\omega)|^2 + |h^{\Delta,2} \cdot h_n^{\Delta,3}(\omega)|^2 - 2\Re \left(h^{\Delta,1} \cdot \overline{h^{\Delta,2}}(\omega) \right) |h_n^{\Delta,3}(\omega)|^2 \right] d\omega. \end{aligned} \quad (3.8)$$

It is easy to see that the first two integrals in Eq. (3.8) are, respectively, the variances of $\sum_{i=1}^n \bar{L}_i^\Delta$ and $L_{n\Delta}$, both equal to $n\Delta$. Setting $n := \lfloor t/\Delta \rfloor$ yields for fixed positive t , as $\Delta \downarrow 0$,

$$\begin{aligned} \text{var} \left[\sum_{i=1}^{\lfloor t/\Delta \rfloor} \bar{L}_i^\Delta - L_{\lfloor t/\Delta \rfloor \Delta} \right] &= 2\lfloor t/\Delta \rfloor \Delta - \frac{1}{\pi} \int_{\mathbb{R}} \Re \left(h^{\Delta,1} \cdot \overline{h^{\Delta,2}}(\omega) \right) \left| h_{\lfloor t/\Delta \rfloor}^{\Delta,3}(\omega) \right|^2 d\omega \\ &= 2t(1 + o(1)) - \frac{1}{\pi} \int_{\mathbb{R}} \Re \left(h^{\Delta,1} \cdot \overline{h^{\Delta,2}}(\omega) \right) \left| h_{\lfloor t/\Delta \rfloor}^{\Delta,3}(\omega) \right|^2 d\omega. \end{aligned}$$

Thus, in order to show Eq. (3.4), it remains to prove that

$$\frac{1}{\pi} \int_{\mathbb{R}} \Re \left(h^{\Delta,1} \cdot \overline{h^{\Delta,2}}(\omega) \right) \left| h_{\lfloor t/\Delta \rfloor}^{\Delta,3}(\omega) \right|^2 d\omega = 2t(1 + o(1)) \quad \text{as } \Delta \downarrow 0,$$

which in turn is equivalent to

$$\begin{aligned} \frac{1}{2\pi t} \int_{\mathbb{R}} \sigma \frac{\sqrt{\Delta}}{\sigma_{\Delta}} \frac{1 - \cos(\omega \lfloor t/\Delta \rfloor \Delta)}{1 - \cos(\omega \Delta)} \left[\frac{\sin(\omega \Delta)}{\omega} \Re \left(\frac{\prod_{j=1}^p (1 - e^{\Delta(\lambda_j + i\omega)})}{\Theta_{\Delta}(e^{i\omega \Delta})} \frac{b(-i\omega)}{a(-i\omega)} \right) \right. \\ \left. + \frac{1 - \cos(\omega \Delta)}{\omega} \Im \left(\frac{\prod_{j=1}^p (1 - e^{\Delta(\lambda_j + i\omega)})}{\Theta_{\Delta}(e^{i\omega \Delta})} \frac{b(-i\omega)}{a(-i\omega)} \right) \right] d\omega = 1 + o(1) \quad \text{as } \Delta \downarrow 0. \end{aligned} \quad (3.9)$$

Now, Lemma A.2 shows that the integrand in Eq. (3.9) converges pointwise, for every $\omega \in \mathbb{R} \setminus \{0\}$, to $2(1 - \cos(\omega t))/\omega^2$ as $\Delta \downarrow 0$. Since, for sufficiently small Δ , the integrand is dominated by an integrable function (see Lemma A.3), we can apply Lebesgue's Dominated Convergence Theorem and deduce that the LHS of Eq. (3.9) converges, as $\Delta \downarrow 0$, to

$$\frac{1}{\pi t} \int_{\mathbb{R}} \frac{1 - \cos(\omega t)}{\omega^2} d\omega = \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos(\omega)}{\omega^2} d\omega = 1.$$

This proves Eq. (3.9) and concludes the proof of the “if”-statement.

As to the “only if”-statement, let $J := \{j = 1, \dots, q : \Re(\mu_j) < 0\}$ and suppose that $|J| \geq 1$. Then we have, by Eq. (2.9) for $\Delta \downarrow 0$,

$$\begin{aligned} \frac{b(-i\omega)}{\Theta_{\Delta}(e^{i\omega \Delta})} &= \prod_{j=1}^{p-q-1} (1 + \eta(\xi_j))^{-1} \prod_{j=1}^q \frac{\mu_j - i\omega}{1 - \zeta_j e^{i\omega \Delta}} \\ &= \prod_{j=1}^{p-q-1} (1 + \eta(\xi_j))^{-1} \Delta^{-q} \prod_{j \in J} \frac{\mu_j - i\omega}{-\mu_j - i\omega} (1 + o(1)) \\ &= \prod_{j=1}^{p-q-1} (1 + \eta(\xi_j))^{-1} \Delta^{-q} (1 + D(\omega))(1 + o(1)), \quad \omega \in \mathbb{R}, \end{aligned} \quad (3.10)$$

where $D(\omega) := -1 + \prod_{j \in J} (\mu_j - i\omega)/(-\mu_j - i\omega)$. By virtue of Lemmata A.2 and A.3, we then obtain that the LHS of Eq. (3.9) converges, as $\Delta \downarrow 0$, to

$$\frac{1}{\pi t} \int_{\mathbb{R}} \frac{1 - \cos(\omega t)}{\omega^2} (1 + \Re(D(\omega))) d\omega = 1 + \frac{1}{\pi} \int_{\mathbb{R}} \frac{1 - \cos(\omega)}{\omega^2} \Re(D(\omega/t)) d\omega.$$

Since $|\prod_{j \in J} (\mu_j - i\omega)/(-\mu_j - i\omega)| = 1$, we further deduce that $\Re(D(\omega)) \leq 0$ for any $\omega \in \mathbb{R}$. Obviously, $\Re(D(\omega)) \not\equiv 0$ and hence,

$$\frac{1}{\pi t} \int_{\mathbb{R}} \frac{1 - \cos(\omega t)}{\omega^2} (1 + \Re(D(\omega))) d\omega < 1,$$

which, in turn, shows that the convergence result (3.4) cannot hold. □

Remark 3.3. (i) It is an easy consequence of the triangle and Hölder's inequality that, if the recovery result (3.4) holds, then also

$$\sum_{i=1}^{\lfloor t/\Delta \rfloor} \bar{L}_i^{\Delta} \sum_{j=\lfloor t/\Delta \rfloor + 1}^{\lfloor s/\Delta \rfloor} \bar{L}_j^{\Delta} \xrightarrow{L^1} L_t(L_s - L_t), \quad t, s \in (0, \infty), \quad t \leq s,$$

is valid.

- (ii) Minor modifications of the proof above show that the recovery result in Eq. (3.4) remains still valid if we drop the assumption of causality (i.e. Assumption 1 (i)) and suppose instead only $\Re(\lambda_j) \neq 0$ for every $j = 1, \dots, p$. Hence, invertibility of the CARMA process is necessary for the noise recovery result to hold, whereas causality is not. In the non-causal case, the obtained white noise process will not be the same as in the Wold representation (3.1).
- (iii) The necessity and sufficiency of the invertibility assumption descends directly from the fact that we always choose the minimum-phase spectral factor, as pointed out in Remark 2.1.
- (iv) The proof suggests that this procedure should work in a much more general framework. Let $I^\Delta(\cdot)$ denote the inversion filter in Eq. (3.5) and $\psi^\Delta := \{\psi_i^\Delta\}_{i \in \mathbb{N}}$ the coefficients in the Wold representation (3.1). Then the proof of Theorem 3.2 essentially needs, apart from the rather technical Lemma A.3, that, as $\Delta \downarrow 0$,

$$I^\Delta(e^{i\omega\Delta})\mathcal{F}\{g(\cdot)\}(\omega) = \frac{\int_0^\infty g(s)e^{i\omega s}ds}{\sum_{k=0}^\infty \psi_k^\Delta e^{ik\omega\Delta}} \rightarrow 1, \quad \omega \in \mathbb{R}, \quad (3.11)$$

provided that the function $\sum_{k=0}^\infty \psi_k^\Delta z^k$ does not have any zero inside the unit circle. In other words, we need that the discrete Fourier transform in the denominator of Eq. (3.11) converges to the Fourier transform in the numerator; this can be easily related to the kernel estimation result in Eq. (3.2). Given the peculiar structure of CARMA processes, this relationship can be calculated explicitly, but the results should hold true for continuous-time moving average processes with more general kernels, too.

We illustrate Theorem 3.2 and the necessity of the invertibility assumption by an example where the convergence result is established using a time domain approach. That gives an explicit result also when the invertibility assumption is violated.

Unfortunately this strategy is not viable for a general CARMA process, due to the complexity of calculations involved when $p > 2$.

Example 3.4 (CARMA(2, q) process). *The causal CARMA(2, q) process is the strictly stationary solution to the formal stochastic differential equation*

$$\begin{aligned} (D - \lambda_2)(D - \lambda_1)Y_t &= \sigma DL_t, & q = 0, \\ (D - \lambda_2)(D - \lambda_1)Y_t &= \sigma(b + D)DL_t, & q = 1, \end{aligned}$$

and it can be represented as a continuous-time moving average process, as in Eq. (2.4), for $\lambda_1, \lambda_2 < 0$, $\lambda_1 \neq \lambda_2$ and $b \in \mathbb{R} \setminus \{0\}$, with kernel

$$\begin{aligned} g(t) &= \sigma \frac{e^{t\lambda_1} - e^{t\lambda_2}}{\lambda_1 - \lambda_2}, & q = 0, \\ g(t) &= \sigma \frac{b + \lambda_1}{\lambda_1 - \lambda_2} e^{t\lambda_1} + \sigma \frac{b + \lambda_2}{\lambda_2 - \lambda_1} e^{t\lambda_2}, & q = 1, \end{aligned}$$

for $t > 0$ and 0 elsewhere. The corresponding sampled process $Y_n^\Delta = Y_{n\Delta}$, $n \in \mathbb{Z}$, satisfies the causal and invertible ARMA(2, 1) equations, as in (2.8) where, from Eq. (27) of Brockwell et al. (2012b) and for $n \in \mathbb{Z}$,

$$\Phi_\Delta(B)Y_n^\Delta = \int_{(n-1)\Delta}^{n\Delta} g(n\Delta - u)dL_u + \int_{(n-2)\Delta}^{(n-1)\Delta} [g(n\Delta - u) - (e^{\lambda_1\Delta} + e^{\lambda_2\Delta})g((n-1)\Delta - u)]dL_u. \quad (3.12)$$

The corresponding MA(1) polynomial in Eq. (2.8) is $\Theta_\Delta(B) = 1 - \theta_\Delta B$, with asymptotic parameters

$$\begin{aligned} \theta_\Delta &= \sqrt{3} - 2 + o(1), & \sigma_\Delta^2 &= \sigma^2 \Delta^3 (2 + \sqrt{3})/6 + o(\Delta^3), & q &= 0, \\ \theta_\Delta &= 1 - \operatorname{sgn}(b) b \Delta + o(\Delta), & \sigma_\Delta^2 &= \sigma^2 \Delta + o(\Delta), & q &= 1. \end{aligned}$$

The inversion of Eq. (2.8) and Eq. (3.12) gives, for every $\Delta > 0$,

$$\begin{aligned} Z_n^\Delta &= \frac{\Phi_\Delta(B)}{\Theta_\Delta(B)} Y_n^\Delta = \sum_{i=0}^{\infty} (\theta_\Delta B)^i \prod_{i=1}^2 (1 - e^{\lambda_i \Delta} B) Y_n^\Delta, \\ &= \int_{(n-1)\Delta}^{n\Delta} g(n\Delta - u) dL_u \\ &\quad + \sum_{i=0}^{\infty} \theta_\Delta^i \int_{(n-i-2)\Delta}^{(n-i-1)\Delta} [g((n-i)\Delta - u) - (e^{\lambda_1 \Delta} + e^{\lambda_2 \Delta} - \theta_\Delta)g((n-i-1)\Delta - u)] dL_u. \end{aligned}$$

Then $Z^\Delta := \{Z_n^\Delta\}_{n \in \mathbb{Z}}$ is a weak white-noise process. Moreover, using that $\Delta L_n = \int_{(n-1)\Delta}^{n\Delta} dL_s$, we have

$$\mathbb{E}[Z_n^\Delta \Delta L_{n-j}] = \begin{cases} 0, & j < 0, \\ \int_0^\Delta g(s) ds, & j = 0, \\ \theta_\Delta^{j-1} \int_0^\Delta [g(\Delta + s) - (e^{\lambda_1 \Delta} + e^{\lambda_2 \Delta} - \theta_\Delta)g(s)] ds, & j > 0. \end{cases} \quad (3.13)$$

For a fixed $t \in (0, \infty)$, using the fact that ΔL and \bar{L}^Δ are both second-order stationary white noises with variance Δ , we have that

$$\begin{aligned} \mathbb{E} \left[\sum_{i=1}^{\lfloor t/\Delta \rfloor} (\bar{L}_i^\Delta - \Delta L_i) \right]^2 &= 2\lfloor t/\Delta \rfloor \Delta - 2 \sum_{i=1}^{\lfloor t/\Delta \rfloor} \mathbb{E}[\bar{L}_i^\Delta \Delta L_i] - 2 \sum_{i \neq j} \mathbb{E}[\bar{L}_i^\Delta \Delta L_j] \\ &= 2\lfloor t/\Delta \rfloor \Delta - \frac{2\sqrt{\Delta}}{\sigma_\Delta} \lfloor t/\Delta \rfloor \int_0^\Delta g(s) ds \\ &\quad - \frac{2\sqrt{\Delta}}{\sigma_\Delta} \int_0^\Delta [g(\Delta + s) - (e^{\lambda_1 \Delta} + e^{\lambda_2 \Delta} - \theta_\Delta)g(s)] ds \sum_{i=1}^{\lfloor t/\Delta \rfloor} \sum_{j=1}^{i-1} \theta_\Delta^{j-1} \end{aligned}$$

where the last equality is obtained using Eq. (3.13). Moreover, for every $a \neq 1$,

$$\sum_{i=1}^n \sum_{j=1}^{i-1} a^{j-1} = \frac{a^n + (1-a)n - 1}{(1-a)^2}, \quad n \in \mathbb{N}.$$

Then the variance of the error can be explicitly calculated as

$$\begin{aligned} \mathbb{E} \left[\sum_{i=1}^{\lfloor t/\Delta \rfloor} (\bar{L}_i^\Delta - \Delta L_i) \right]^2 &= 2\lfloor t/\Delta \rfloor \Delta - \frac{2\sqrt{\Delta}}{\sigma_\Delta} \lfloor t/\Delta \rfloor \int_0^\Delta g(s) ds \\ &\quad - \frac{2\sqrt{\Delta} \theta_\Delta^{\lfloor t/\Delta \rfloor} + \lfloor t/\Delta \rfloor (1 - \theta_\Delta) - 1}{\sigma_\Delta (1 - \theta_\Delta)^2} \int_0^\Delta [g(\Delta + s) - (e^{\lambda_1 \Delta} + e^{\lambda_2 \Delta} - \theta_\Delta)g(s)] ds. \end{aligned}$$

We now compute the asymptotic expansion for $\Delta \downarrow 0$ of the equation above. We obviously have that $2\lfloor t/\Delta \rfloor \Delta = 2t(1 + o(1))$ and, using the explicit formulas for the kernel functions g ,

$$\begin{aligned} \frac{2\sqrt{\Delta}}{\sigma_{\Delta}} \lfloor t/\Delta \rfloor \int_0^{\Delta} g(s) ds &= \begin{cases} \frac{q=0}{(3-\sqrt{3})} t + o(1), \\ \frac{q=1}{2t + o(1)}, \end{cases} \\ \frac{2\sqrt{\Delta}}{\sigma_{\Delta}} \int_0^{\Delta} [g(\Delta + s) - (e^{\lambda_1 \Delta} + e^{\lambda_2 \Delta} - \theta_{\Delta})g(s)] ds &= \begin{cases} (4\sqrt{3} - 6) \Delta(1 + o(1)), \\ 2(b - \operatorname{sgn}(b)b)\Delta^2 + o(\Delta^2), \end{cases} \\ (\theta_{\Delta}^{\lfloor t/\Delta \rfloor} + \lfloor t/\Delta \rfloor(1 - \theta_{\Delta}) - 1)(1 - \theta_{\Delta})^{-2} &= \begin{cases} \frac{1}{6} (3 + \sqrt{3}) t/\Delta(1 + o(1)), \\ (e^{-\operatorname{sgn}(b)bt} + \operatorname{sgn}(b)bt - 1)/(b\Delta)^2 + o(\Delta^{-2}). \end{cases} \end{aligned}$$

Hence, for a fixed $t \in (0, \infty)$ and $\Delta \downarrow 0$, we get

$$\mathbb{E} \left[\sum_{i=1}^{\lfloor t/\Delta \rfloor} (\bar{L}_i^{\Delta} - \Delta L_i) \right]^2 = \begin{cases} o(1), & q = 0, \\ 2(e^{-\operatorname{sgn}(b)bt} + \operatorname{sgn}(b)bt - 1)(\operatorname{sgn}(b) - 1)/b + o(1), & q = 1, \end{cases}$$

i.e. (3.4) holds always for $q = 0$, whereas for $q = 1$ if and only if $b > 0$. If $b < 0$, the error approximating the driving Lévy by inversion of the discretized process grows as $4t$ for large t .

4 High-frequency behaviour of approximating Riemann sums

The fact that, in the sense of Eq. (3.3), $\bar{L}_n^{\Delta} \approx \Delta L_n = L_{n\Delta} - L_{(n-1)\Delta}$ for small Δ , along with Eq. (3.2), gives rise to the conjecture that the Wold representation Eq. (3.1) for Y^{Δ} behaves on a high-frequency time grid approximately like the $\text{MA}(\infty)$ process

$$\tilde{Y}_n^{\Delta, h} := \sum_{j=0}^{\infty} g(\Delta(j+h)) \Delta L_{n-j}, \quad n \in \mathbb{Z}, \quad (4.1)$$

for some $h \in [0, 1]$ and g is the kernel function as in (2.6). In other terms, we have for a CARMA process, under the assumption of invertibility and causality, that the discrete-time quantities appearing in the Wold representation approximate the quantities in Eq. (4.1) when $\Delta \downarrow 0$. The transfer function of Eq. (4.1) is then defined as

$$\psi_h^{\Delta}(\omega) := \sum_{j=0}^{\infty} g(\Delta(j+h)) e^{-i\omega j}, \quad -\pi \leq \omega \leq \pi, \quad (4.2)$$

and its spectral density can be written as

$$\tilde{f}_h^{\Delta}(\omega) = \frac{1}{2\pi} |\psi_h^{\Delta}|^2(\omega), \quad -\pi \leq \omega \leq \pi.$$

It is well known that a CMA process can be defined (for a fixed time point t) as the L^2 -limit of Eq. (4.1); this fact is naturally employed to easily simulate a CMA process, when all the relevant quantities are known a priori. Therefore we will name the process $\tilde{Y}^{\Delta, h}$ *approximating Riemann sum* of Eq. (2.4), and h is the so-called *rule* of the approximating sum; e.g. if $h = 1/2$, we have the popular *mid-point rule*.

In order to give an answer to our conjecture, we will investigate properties of the approximating Riemann sum $\tilde{Y}^{\Delta, h}$ of a CARMA process and compare its asymptotic autocovariance structure with the one of the sampled CARMA process Y^{Δ} . This will yield more insight into the role of h for the behavior of $\tilde{Y}^{\Delta, h}$ as a process.

We start with a well known property of approximating sums.

Proposition 4.1. *Let g be in L^2 and Riemann-integrable. Then, for every $h \in [0, 1]$, as $\Delta \downarrow 0$:*

$$(i) \quad \tilde{Y}_k^{\Delta, h} - Y_k^\Delta \xrightarrow{L^2} 0, \text{ for every } k \in \mathbb{Z}.$$

$$(ii) \quad \tilde{Y}_{\lfloor t/\Delta \rfloor}^{\Delta, h} \xrightarrow{L^2} Y_t, \text{ for every } t \in \mathbb{R}.$$

Proof. This follows immediately from the hypotheses made on g and the definition of L^2 -integrals. \square

This result essentially says only that approximating sums converge to Y_t for every fixed t . However, for a CARMA(p, q) process we have that the approximating Riemann sum process satisfies for every h and Δ an ARMA($p, p-1$) equation (see Proposition 4.2 below), meaning that there might exist a process, whose autocorrelation structure is the same as the one of the approximating sum. Given that the AR filter in this representation is the same as in Eq. (2.8), it is reasonable to investigate whether $\Phi_\Delta(B)Y^\Delta$ and $\Phi_\Delta(B)\tilde{Y}^{\Delta, h}$ have, as $\Delta \downarrow 0$, the same asymptotic autocovariance structure, which can be expected but is not granted by Proposition 4.1.

The following proposition gives the ARMA($p, p-1$) representation for the approximating Riemann sum.

Proposition 4.2. *Let Y be a CARMA(p, q) process, satisfying Assumption 1, and furthermore suppose that the roots of the autoregressive polynomial $a(\cdot)$ are distinct. Then the approximating Riemann sum process $\tilde{Y}^{\Delta, h}$ of Y defined by Eq. (4.1) satisfies, for every $h \in [0, 1]$, the ARMA($p, p-1$) equation*

$$\Phi_\Delta(B)\tilde{Y}_n^{\Delta, h} = \sigma\tilde{\Theta}_{\Delta, h}(B)\Delta L_n, \quad n \in \mathbb{Z}, \quad (4.3)$$

where

$$\tilde{\Theta}_{\Delta, h}(z) := \tilde{\theta}_0^{\Delta, h} - \tilde{\theta}_1^{\Delta, h}z + \dots + (-1)^{p-1}\tilde{\theta}_{p-1}^{\Delta, h}z^{p-1} \quad (4.4)$$

and

$$\tilde{\theta}_k^{\Delta, h} := \sum_{l=1}^p \frac{b(\lambda_l)}{a'(\lambda_l)} e^{h\Delta\lambda_l} \sum e^{\Delta(\lambda_{j_1} + \lambda_{j_2} + \dots + \lambda_{j_k})}, \quad k = 0, \dots, p-1,$$

where the right-hand sum is defined to be 1 for $k = 0$ and it is evaluated over all possible subsets $\{j_1, \dots, j_k\}$ of $\{1, \dots, p\} \setminus \{l\}$, having cardinality k , for $k > 0$.

Proof. Write $\Phi_\Delta(z) = \prod_{j=1}^p (1 - e^{\Delta\lambda_j}z) = -\sum_{j=0}^p \phi_j^\Delta z^j$ and observe that

$$\begin{aligned} \Phi_\Delta(B)\tilde{Y}_n^{\Delta, h} &= -\sum_{j=0}^p \phi_j^\Delta Y_{n-j}^{\Delta, h} \\ &= -\sigma \mathbf{b}^\top \sum_{k=0}^{p-1} \left(\sum_{j=0}^k \phi_j^\Delta e^{A(k-j)\Delta} \right) e^{Ah\Delta} \mathbf{e}_p \cdot \Delta L_{n-k} \\ &\quad - \sigma \mathbf{b}^\top \sum_{j=0}^p \sum_{k=p-j}^{\infty} \phi_j^\Delta e^{A(h+k)\Delta} \mathbf{e}_p \cdot \Delta L_{n-j-k} \\ &= -\sigma \mathbf{b}^\top \sum_{k=0}^{p-1} \left(\sum_{j=0}^k \phi_j^\Delta e^{A(k-j)\Delta} \right) e^{Ah\Delta} \mathbf{e}_p \cdot \Delta L_{n-k} \end{aligned}$$

$$-\sigma \mathbf{b}^\top \sum_{k=p}^{\infty} \left(-\sum_{j=0}^p \phi_j^\Delta e^{-Aj\Delta} \right) e^{A(h+k)\Delta} \mathbf{e}_p \cdot \Delta L_{n-k}.$$

By virtue of the Cayley-Hamilton theorem (cf. also (Brockwell and Lindner, 2009, proof of Lemma 2.1)), we have that

$$-\sum_{j=0}^p \phi_j^\Delta e^{-Aj\Delta} = 0,$$

and hence, $\Phi_\Delta(B) \tilde{Y}_n^{\Delta,h} = -\sigma \mathbf{b}^\top \sum_{k=0}^{p-1} \left(\sum_{j=0}^k \phi_j^\Delta e^{A(k-j)\Delta} \right) e^{Ah\Delta} \mathbf{e}_p \cdot \Delta L_{n-k}$. We conclude with (Fasen and Fuchs, 2012, Lemma 2.1(i) and Eq. (4.4)). \square

Remark 4.3. (i) *The approximating Riemann sum of a causal CARMA process is automatically a causal ARMA process. On the other hand, even if the CARMA process is invertible in the sense of Definition 3.1, the roots of $\tilde{\Theta}_{\Delta,h}(\cdot)$ may lie inside the unit circle, causing $\tilde{Y}^{\Delta,h}$ to be non-invertible.*

(ii) *It is easy to see that $\tilde{\theta}_0^{\Delta,h} = g(h\Delta)$. Then for $p - q \geq 2$, if $h = 0$, we have that $\tilde{\theta}_0^{\Delta,0} = 0$, giving that $\tilde{\Theta}_{\Delta,0}(0) = 0$. This is never the case for $\Theta_\Delta(\cdot)$, as one can see from Eq. (2.9) and Proposition 2.2. Moreover, it is possible to show that for $h = 1$ and $p - q \geq 2$, the coefficient $\tilde{\theta}_{p-1}^{\Delta,1}$ is 0, implying that Eq. (4.3) is actually an ARMA($p, p - 2$) equation. For those values of h , the ARMA equations solved by the approximating Riemann sums will never have the same asymptotic form as Eq. (2.8): therefore we shall restrict ourselves to the case $h \in (0, 1)$ from now on.*

(iii) *The assumption of distinct autoregressive roots might seem restrictive, but the omitted cases can be obtained by letting distinct roots tend to each other. This would, of course, change the coefficients of the MA polynomial in Eq. (4.4). Moreover, as shown in Brockwell et al. (2012a,b), the multiplicity of the zeroes does not matter when L^2 -asymptotic relationships as $\Delta \downarrow 0$ are considered.*

Due to the complexity of retrieving the roots of a polynomial of arbitrary order from its coefficients, finding the asymptotic expression of $\tilde{\Theta}_{\Delta,h}(\cdot)$ for arbitrary p is a daunting task. Nonetheless, using Proposition 4.2, it is not difficult to give an answer for processes with $p \leq 3$, which are the most used in practice.

Proposition 4.4. *Let $\tilde{Y}^{\Delta,h}$ be the approximating Riemann sum for a CARMA(p, q) process, with $p \leq 3$, let Assumption 1 hold and the roots of $a(\cdot)$ be distinct.*

If $p = 1$, the process $\tilde{Y}^{\Delta,h}$ is an AR(1) process driven by $Z_n^\Delta = \sigma e^{\Delta h \lambda_1} \Delta L_n$, for every $\Delta > 0$. If $p = 2, 3$, we have

$$\Phi_\Delta(B) \tilde{Y}_n^{\Delta,h} = \prod_{i=1}^q (1 - (1 - \Delta \mu_i + o(\Delta))B) \prod_{i=1}^{p-q-1} (1 - \chi_{p-q,i}(h)B) \left(\sigma \frac{(h\Delta)^{p-q-1}}{(p-q-1)!} \Delta L_n \right) \quad (4.5)$$

where, for $h \in (0, 1)$ and $\Delta \downarrow 0$, $\chi_{2,1}(h) = (h - 1)/h + o(1)$ and

$$\chi_{3,j}(h) = \frac{2(h-1)^2}{2(h-1)h - 1 - (-1)^j \sqrt{1 - 4(h-1)h}} + o(1), \quad j = 1, 2.$$

Proof. Since $p \leq 3$, $\tilde{\Theta}_{\Delta,h}(z)$ is a polynomial of order $p-1$ with real coefficients; therefore the roots, if any, can be calculated from the coefficients, and asymptotic expressions can be obtained by computing the Taylor expansions of the roots around $\Delta = 0$.

If $p = 1$, the statement follows directly from Eq. (4.3). For $p = 2, 3$, the roots of Eq. (4.4) are $\{1 + \Delta\mu_i + o(\Delta)\}_{i=1,\dots,q}$ and $\{1/\chi_{p-q,i}(h)\}_{i=1,\dots,p-q-1}$, giving that

$$\tilde{\Theta}_{\Delta,h}(z) = \tilde{\theta}_{p-1}^{\Delta,h} \prod_{i=1}^q (1 + \Delta\mu_i + o(\Delta) - z) \prod_{i=1}^{p-q-1} (1/\chi_{p-q,i}(h) - z), \quad z \in \mathbb{C}.$$

Vieta's theorem gives that the product of the roots of Eq. (4.4) is equal to $\tilde{\theta}_0^{\Delta,h}/\tilde{\theta}_{p-1}^{\Delta,h}$, which yields

$$\tilde{\Theta}_{\Delta,h}(z) = \tilde{\theta}_0^{\Delta,h} \prod_{i=1}^q (1 - (1 - \Delta\mu_i + o(\Delta))z) \prod_{i=1}^{p-q-1} (1 - \chi_{p-q,i}(h)z).$$

Since $\tilde{\theta}_0^{\Delta,h} = g(h\Delta) = \sigma(h\Delta)^{p-q-1}/(p-q-1)!(1+o(1))$, we have established the result. \square

In general the autocorrelation structure will depend on h through the parameters $\chi_{p-q,i}(h)$, $i = 1, \dots, p-q-1$. In a time-series context, it is reasonable to require that the approximating Riemann sum process has the same asymptotic autocorrelation structure as the CARMA process we want to approximate.

Corollary 4.5. *Let the assumptions of Proposition 4.4 hold. Then $\Phi_{\Delta}(B)Y^{\Delta}$ and $\Phi_{\Delta}(B)\tilde{Y}^{\Delta,h}$ have the same asymptotic autocovariance structure as $\Delta \downarrow 0$*

$$\begin{aligned} &\text{for every } h \in (0, 1), && \text{if } p - q = 1, \\ &\text{for } h = (3 \pm \sqrt{3})/6, && \text{if } p - q = 2, \\ &\text{and for } h = \left(15 \pm \sqrt{225 - 30\sqrt{30}}\right)/30, && \text{if } p - q = 3. \end{aligned}$$

Moreover, the MA polynomials in Eqs. (2.9) and (4.5) coincide if and only if the CARMA process is invertible and $|\chi_{p-q,i}(h)| < 1$, that is

$$\begin{aligned} &\text{for every } h \in (0, 1), && \text{if } p - q = 1, \\ &\text{for } h = (3 + \sqrt{3})/6, && \text{if } p - q = 2. \end{aligned}$$

For $p - q = 3$, such h does not exist.

Proof. The claim for $p - q = 1$ follows immediately from Proposition 4.4 and Eqs. (2.9)-(2.10). For $p = 2$ and $q = 0$, we have to solve the spectral factorization problem

$$\begin{aligned} \sigma_{\Delta}^2(1 + \eta(\xi_1)^2) &= \sigma^2\Delta^3(1 + \chi_{2,1}(h)^2)h^2 \\ \sigma_{\Delta}^2\eta(\xi_1) &= -\sigma^2\Delta^3\chi_{2,1}(h)h^2 \end{aligned}$$

with $\eta(\xi_1) = 2 - \sqrt{3} + o(1)$ and $\chi_{2,1}(h) = (h-1)/h + o(1)$. Eq. (2.10) then yields the two solutions $h = (3 \pm \sqrt{3})/6$. For $p = 3$ and $q = 1$, analogous calculations lead to the same solutions. Finally, consider the case $p = 3$ and $q = 0$. We have to solve asymptotically the following system of equations

$$\sigma_{\Delta}^2(1 + (\eta(\xi_1) + \eta(\xi_2))^2 + \eta(\xi_1)^2\eta(\xi_2)^2) = \frac{\sigma^2\Delta^5}{4}(1 + (\chi_{3,1}(h) + \chi_{3,2}(h))^2 + \chi_{3,1}(h)^2\chi_{3,2}(h)^2)h^4$$

$$\begin{aligned}\sigma_\Delta^2(\eta(\xi_1) + \eta(\xi_2))(1 + \eta(\xi_1)\eta(\xi_2)) &= -\frac{\sigma^2\Delta^5}{4}(\chi_{3,1}(h) + \chi_{3,2}(h))(1 + \chi_{3,1}(h)\chi_{3,2}(h))h^4 \\ \sigma_\Delta^2\eta(\xi_1)\eta(\xi_2) &= \frac{\sigma^2\Delta^5}{4}\chi_{3,1}(h)\chi_{3,2}(h)h^4\end{aligned}$$

where $\eta(\xi_{1,2}) = (13 \pm \sqrt{105} - \sqrt{270 \pm 26\sqrt{105}})/2 + o(1)$ and $\chi_{3,1}(h)$ and $\chi_{3,2}(h)$ are as in Proposition 4.4. Solving that system for h gives the claimed values.

To prove the second part of the Corollary, we start observing that, under the assumption of an invertible CARMA process, the coefficients depending on μ_i will coincide automatically, if any. Then it remains to check whether the coefficients depending on h can be smaller than 1 in absolute value. The cases $p - q = 1, 2$ follow immediately. Moreover, to see that there is no such h for $p - q = 3$, it is enough to notice that, for $h \in (0, 1)$, $|\chi_{3,1}(h)| > 1$ and $0 < |\chi_{3,2}(h)| < 1$, i.e. they do never satisfy the sought requirement for $h \in (0, 1)$. \square

Corollary 4.5 can be interpreted as a criterion to choose an h such that the Riemann sum approximates the continuous-time process Y in a stronger sense than the simple convergence as a random variable for every fixed t . The second part of the corollary says that there is an even more restrictive way to choose h such that Eqs. (2.9) and (4.5) will coincide. If the two processes satisfy asymptotically the same causal and invertible ARMA equation, they will have the same coefficients in their Wold representations as $\Delta \downarrow 0$, which are given in the case of the approximating Riemann sum explicitly by definition in Eq. (4.1).

In the light of Eq. (3.2) and Theorem 3.2, the sampled CARMA process will asymptotically behave like its approximating Riemann sum process for some specific $h = \bar{h}$, which might not even exist, as in the case $p = 3, q = 0$. However, if such an \bar{h} exists, the kernel estimators (3.2) can be improved to

$$\frac{\sigma_\Delta}{\sqrt{\Delta}}\psi_{[t/\Delta]}^\Delta = g(\Delta([t/\Delta] + \bar{h})) + o(\Delta), \quad t \in \mathbb{R}.$$

For invertible CARMA(p, q) processes with $p - q = 1$, any choice of h would accomplish that. In principle an \bar{h} can be found matching a higher order expansion in Δ , where higher order terms will depend on h .

For $p - q = 2$, there is only a specific value $h = \bar{h} := (3 + \sqrt{3})/6$, such that $\tilde{Y}^{\Delta, \bar{h}}$ behaves as Y^Δ in this particular sense, and therefore it advocates for a unique, optimal value for, e.g., simulation purposes.

Finally, for $p - q = 3$, a similar value does not exist, meaning that it is not possible to mimic Y^Δ in this sense with any approximating Riemann sum.

In order to confirm this, we now give a small numerical study. We consider three different causal and invertible processes, a CARMA(2, 1), a CAR(2) and a CAR(3) process, with parameters $\lambda_1 = -0.7, \lambda_2 = -1.2, \lambda_3 = -2.6$ and $\mu_1 = 3$. Of course, for the CARMA(2, 1) we use only λ_1, λ_2 and μ_1 , while for the CAR processes there is no need for μ_1 . Then we estimate the kernel functions from the theoretical autocorrelation functions using (3.2) as in Brockwell et al. (2012a), for moderately high sampling rates, namely $2^2 = 4$ (Figure 1) and $2^6 = 64$ samplings per unit of time (Figure 2). In order to see where the kernel is being estimated, we plot the kernel estimations on different grids. The small circles denote the extremal cases $h = 0$ and $h = 1$, the vertical sign the mid-point rule $h = 0.5$, and the diamond and the square are the values given in Corollary 4.5, if any. The true kernel function is then plotted with a solid, continuous line. For the sake of clarity, only the first 8 estimates are plotted.

For the CARMA(2,1) process, the kernel estimation seems to follow a mid-point rule (i.e. $h = 1/2$). For the CAR(2) process, the predicted value $\bar{h} = (3 + \sqrt{3})/6$ (denoted with squares) is definitely the correct one, and for the CAR(3) for every $h \in [0, 1]$ the estimation is close but constantly biased. Of course in the limit $\Delta \downarrow 0$, the slightly weaker results given by Eq. (3.2) still hold, giving that the bias vanishes in the limit. The conclusion expressed above is true for both considered sampling rates, which is remarkable since they are only moderately high, in comparison with the chosen parameters.

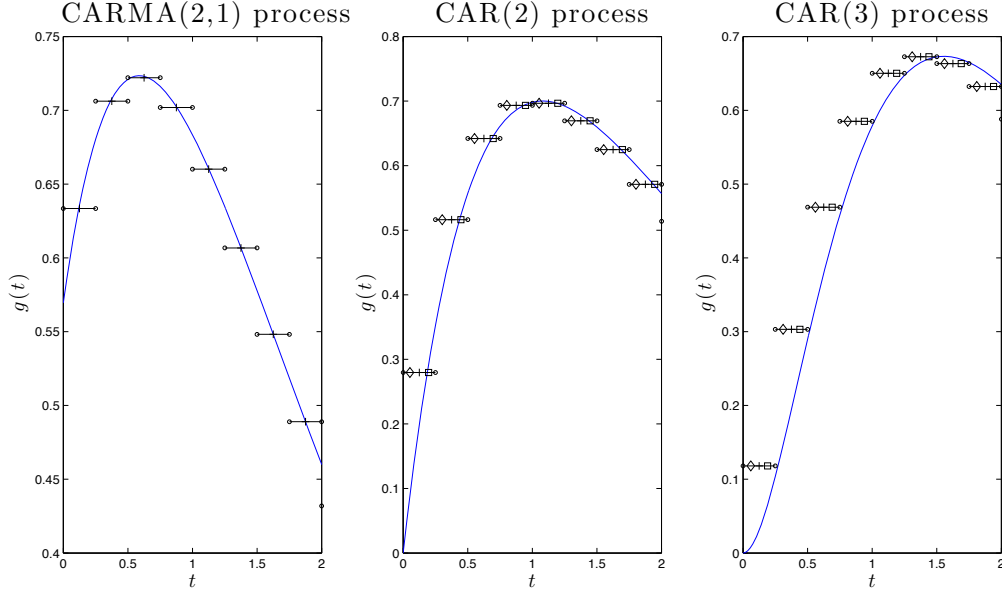


Figure 1: Kernel estimation for a sampling frequency of 2^2 samplings per unit of time, i.e. $\Delta = 0.25$. The diamond and the square symbols denote, where available, the values of h suggested by Corollary 4.5.

5 Conclusion

In this paper we have dealt with the asymptotic behaviour of two classes of discrete-time processes closely related to the one of CARMA processes, when the sampling frequency tends to infinity.

First, we studied the behaviour of the white noise appearing in the ARMA equation solved by the sampled sequence of a CARMA process of arbitrary order. Then we showed, under a necessary and sufficient identifiability condition, that the aforementioned white noise approximates the increments of the driving Lévy process of the continuous-time model. The proposed procedure is non-parametric in nature, and it can be applied without assuming any knowledge on the order of the process. Moreover, it is argued that such results should be extendable to CMA processes with more general kernel functions. This is left for future research.

The results in the first part of this paper, considered jointly with those in Brockwell et al. (2012a), show that the Wold representation of a sampled causal and invertible CARMA process behaves somehow like a Riemann sum as $\Delta \downarrow 0$. Then in the second part of this paper the converse is investigated, i.e. whether a Riemann sum approximating a causal CARMA process satisfies

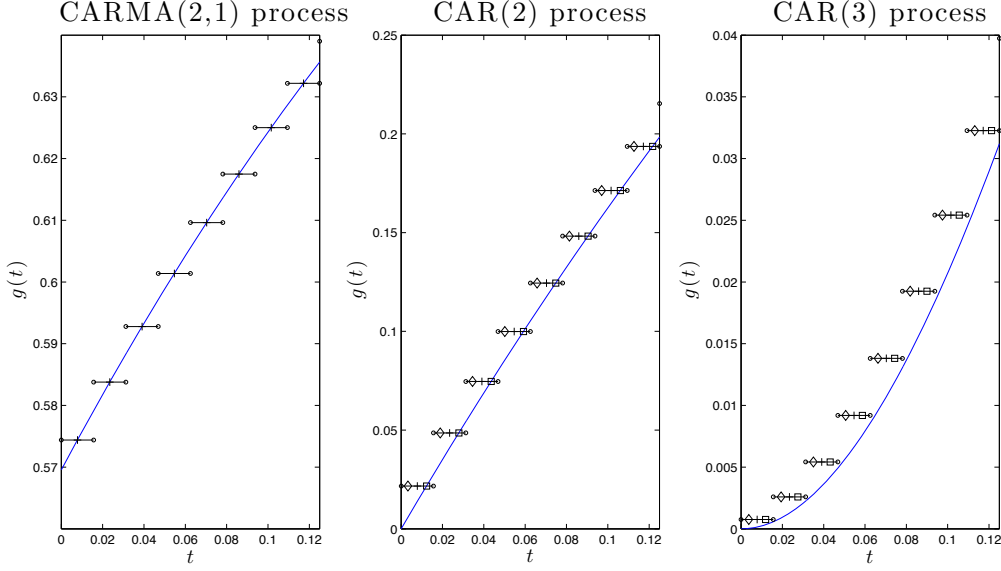


Figure 2: Kernel estimation for a sampling frequency of 2^6 samplings per unit of time, i.e. $\Delta \approx 0.016$. The diamond and the square symbols denote, where available, the values of h suggested by Corollary 4.5.

the same $\text{ARMA}(p, p-1)$ equation of the process we want to approximate, as the spacing of the grid tends to zero. This is in particular important for simulations, where such approximating Riemann sums constitute a natural choice.

It has been shown that these processes satisfy an $\text{ARMA}(p, p-1)$ equation, but in general they are not invertible. For $p \leq 3$, the roots of the moving average polynomial of such discrete-time processes depend, apart from the roots of $b(\cdot)$, also on the rule h . Moreover, in the case $p = 3$, $q = 0$, the Riemann sum is invertible for no choice of h , implying that the Riemann sum and the sampled process will never satisfy asymptotically the same causal and invertible ARMA equation. Although only a finite number of cases has been considered, it shows that in general a causal and invertible CARMA process cannot be approximated by a Riemann sum, at least not in the sense of the second part of Corollary 4.5. Further investigations on this matter are left for future research.

A Auxiliary Results

Throughout the appendix, we shall use the same notation as in the preceding sections.

Proposition A.1. *The function $\alpha_n(x)$, as defined in Eq. (2.12), has the form*

$$\alpha_n(x) = \frac{1}{(2n+1)!} \frac{P_n(x)}{x^{n+1}}, \quad x \neq 0, \quad n \in \mathbb{N}, \quad (\text{A.1})$$

where $P_n(x)$ is a polynomial of order n in x , namely

$$P_n(x) = \sum_{j=0}^n x^{n-j} \sum_{k=j+1}^n (2k)! \left\{ \begin{matrix} 2n+1 \\ 2k \end{matrix} \right\} \sum_{i=j}^k \left[\binom{i+1}{j+1} \binom{2k}{2i+1} - \binom{i}{j+1} \binom{2k}{2i} \right] (-2)^{j+1-2k}$$

$$+ \sum_{j=0}^n x^{n-j} \sum_{k=j}^n (2k+1)! \left\{ \begin{matrix} 2n+1 \\ 2k+1 \end{matrix} \right\} \sum_{i=j}^k \left[\binom{i+1}{j+1} \binom{2k+1}{2i+1} - \binom{i}{j+1} \binom{2k+1}{2i} \right] (-2)^{j-2k}, \quad (\text{A.2})$$

with $\left\{ \cdot \right\}$ being the Stirling number of the second kind. Moreover, all the zeroes of $\alpha_n(x)$ are real, distinct and greater than 2.

Proof. Using the definition of the hyperbolic functions, we can write

$$\frac{\sinh(z)}{\cosh(z) - 1 + x} = \frac{e^{2z} - 1}{e^{2z} + 1 + 2(x-1)e^z} =: f(z, x), \quad z \in \mathbb{C}, \quad x \neq 0,$$

i.e. $f(z, x) = g(e^z, x)$, where $g(y, x) := (y^2 - 1)/(y^2 + 1 + 2(x-1)y)$. Then the n -th derivative of the function $f(\cdot, x)$ is, by the Faà di Bruno formula

$$\frac{\partial^n}{\partial z^n} f(z, x) = \frac{\partial^n}{\partial z^n} g(e^z, x) = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{\partial^k}{\partial y^k} g(y, x) \Big|_{y=e^z} e^{kz}, \quad z \in \mathbb{C}, \quad x \neq 0,$$

where the coefficients $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ are the Stirling numbers of the second kind. Using the previous formula and the definition of $\alpha_n(x)$, for $x \neq 0$,

$$\begin{aligned} (2n+1)! \alpha_n(x) &= \frac{\partial^{2n+1}}{\partial z^{2n+1}} f(z, x) \Big|_{z=0} = \sum_{k=1}^{2n+1} \left\{ \begin{matrix} 2n+1 \\ k \end{matrix} \right\} \frac{\partial^k}{\partial y^k} g(y, x) \Big|_{y=1} \\ &= \sum_{k=1}^{2n+1} k! \left\{ \begin{matrix} 2n+1 \\ k \end{matrix} \right\} \frac{1}{2\pi i} \int_{\rho} \frac{(z-1)(z+1)}{(z-a_2(x))(z-a_1(x))(z-1)^{k+1}} dz \\ &= \sum_{k=1}^{2n+1} k! \left\{ \begin{matrix} 2n+1 \\ k \end{matrix} \right\} \frac{1}{2\pi i} \int_{\rho} \frac{z+1}{(z-a_2(x))(z-a_1(x))(z-1)^k} dz \end{aligned} \quad (\text{A.3})$$

where the latter equality comes from the Cauchy differentiation formula, $a_1(x) = 1 - x - \sqrt{x(x-2)}$, $a_2(x) = 1 - x + \sqrt{x(x-2)}$, i.e. they are the roots of the polynomial $y^2 + 1 + 2(x-1)y$, and ρ is a counter-clockwise oriented closed curve in the complex plane encircling $z = 1$. The case $x = 0$ has been excluded because $f(\cdot, 0)$ is not defined in $z = 0$, and $\lim_{z \rightarrow 0} |f(z, 0)| = \infty$.

Let us denote the integrand in Eq. (A.3) by $f_k(z, x)$; it is a rational function having one pole of order k in $z = 1$. Moreover, there are two simple poles in $z = a_1(x)$ and $z = a_2(x)$ if $x \neq 2$, or just a simple one in $z = -1$ if $x = 2$, due to cancellation with the zero at $z = -1$ in the numerator. The case $x = 2$ can be also obtained by letting the difference between $a_1(x)$ and $a_2(x)$ tend to zero in the upcoming calculations and is therefore not treated by itself.

Then by the Cauchy theorem of residues, Theorem 1, p. 24 of Mitrinović and Kečkić (1984), we have that the integral in Eq. (A.3) is

$$\frac{1}{2\pi i} \int_{\rho} f_k(z, x) dz = \text{Res}_{z=1} f_k(z, x) = - \text{Res}_{z=a_1(x)} f_k(z, x) - \text{Res}_{z=a_2(x)} f_k(z, x) - \text{Res}_{z=\infty} f_k(z, x), \quad x \neq 0, 2,$$

where the latter identity is obtained using Theorem 2, p. 25 of Mitrinović and Kečkić (1984). Moreover, since the difference of orders between the polynomials in the denominator and in the numerator of $f_k(z, x)$ is $k+1 > 1$, the residue at infinity is always zero (Section 3.1.2.3, pp.

27-28, Mitrinović and Kečkić (1984)). For $x \neq 0, 2$, we have that $z = a_1(x)$ and $z = a_2(x)$ are simple poles, yielding

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\rho} f_k(z, x) dz &= \frac{1 + a_1(x)}{(a_1(x) - 1)^k (a_2(x) - a_1(x))} - \frac{1 + a_2(x)}{(a_2(x) - 1)^k (a_2(x) - a_1(x))} \\
&= (2x)^{-k} \sum_{i=0}^{\lfloor k/2 \rfloor} \left[2 \binom{k}{2i+1} (-x)^{-1} + \binom{k}{2i+1} - \binom{k}{2i} \right] (-x)^{k-2i} (x(x-2))^i \\
&= \sum_{j=0}^{\lfloor k/2 \rfloor - 1} \sum_{i=j}^{\lfloor k/2 \rfloor} \left[\binom{i+1}{j+1} \binom{k}{2i+1} - \binom{i}{j+1} \binom{k}{2i} \right] (-2)^{j+1-k} x^{-j-1} \quad (\text{A.4}) \\
&\quad + (k \bmod 2) (-2)^{\lfloor k/2 \rfloor + 1 - k} x^{-\lfloor k/2 \rfloor - 1},
\end{aligned}$$

where the last equality is obtained by using the Binomial theorem. Plugging Eq. (A.4) in Eq. (A.3), and dividing the outermost sum in odd and even k , we get, still for $x \neq 0, 2$,

$$\begin{aligned}
(2n+1)! \alpha_n(x) &= \sum_{k=1}^n \sum_{j=0}^{k-1} \sum_{i=j}^k (2k)! \left\{ \begin{matrix} 2n+1 \\ 2k \end{matrix} \right\} \left[\binom{i+1}{j+1} \binom{2k}{2i+1} - \binom{i}{j+1} \binom{2k}{2i} \right] (-2)^{j+1-2k} x^{-j-1} \\
&\quad + \sum_{k=0}^n \sum_{j=0}^{k-1} \sum_{i=j}^k (2k+1)! \left\{ \begin{matrix} 2n+1 \\ 2k+1 \end{matrix} \right\} \left[\binom{i+1}{j+1} \binom{2k+1}{2i+1} - \binom{i}{j+1} \binom{2k+1}{2i} \right] (-2)^{j-2k} x^{-j-1} \\
&\quad + \sum_{k=0}^n (2k+1)! \left\{ \begin{matrix} 2n+1 \\ 2k+1 \end{matrix} \right\} (-2)^{-k} x^{-k-1}.
\end{aligned}$$

Then Eq. (A.1) can be obtained by merging the last two lines and rearranging the indices.

Using Eq. (A.2), we easily see that, for $P_n(x) = p_0 + p_1 x + \dots + p_n x^n$,

$$p_0 = (-2)^{-n} (2n+1)!, \quad p_n = 1, \quad (\text{A.5})$$

i.e. $P_n(x)$ will have n , potentially complex, roots, and they can not be zero. Moreover it is easy to verify that $f(z, x)$ solves the mixed partial differential equation

$$\frac{\partial^2}{\partial z^2} f(z, x) = \left[(x-1) \frac{\partial}{\partial x} + x(x-2) \frac{\partial^2}{\partial x^2} \right] f(z, x), \quad z \in \mathbb{C}, \quad x \neq 0. \quad (\text{A.6})$$

Then we take $2n-1$, $n \in \mathbb{N}$, derivatives in z on both sides of Eq. (A.6); invoking the Schwarz theorem, the product rule for the derivatives and evaluating the resulting expression at $z = 0$, we obtain that the function $\alpha_n(x)$ is given by recursion, for $x \notin (0, 2)$,

$$\begin{aligned}
(2n+3)(2n+1)\alpha_{n+1}(x) &= \sqrt{x(x-2)} \frac{\partial}{\partial x} \left[\sqrt{x(x-2)} \frac{\partial}{\partial x} \alpha_n(x) \right], \quad (\text{A.7}) \\
\alpha_0(x) &= 1/x.
\end{aligned}$$

We now prove by induction the claim regarding the roots being real, distinct and greater than 2. The cases $\alpha_0(x) = 1/x$ and $6\alpha_1(x) = (x-3)/x^2$ have respectively no and one zero, so the claim can only be partially verified; then we start from $\alpha_2(x) = (30 - 15x + x^2)/(120x^3)$, whose zeros are $\xi_{2,1} = 1/2(15 - \sqrt{105}) \approx 2.37652$, $\xi_{2,2} = 1/2(15 + \sqrt{105}) \approx 12.6235$, and they satisfy the

claim. We now assume that the claim holds for $\alpha_n(x)$, $n \geq 2$, and its zeros are $2 < \xi_{n,1} < \xi_{n,2} < \dots < \xi_{n,n}$.

The derivative of $\alpha_n(x)$ is of the form $Q_n(x)/x^{n+2}$, where $(2n+1)!Q_n(x) = x \frac{\partial}{\partial x} P_n(x) - (1+n)P_n(x)$. By Rolle's theorem, $Q_n(x)$ has $n-1$ real roots $\chi_{n,i}$, $i = 1, \dots, n-1$, such that $2 < \xi_{n,1} < \chi_{n,1} < \xi_{n,2} < \chi_{n,2} < \dots < \chi_{n,n-1} < \xi_{n,n}$. Using the product rule and the value of the coefficients in Eq. (A.5), we get

$$\frac{\partial}{\partial x} \alpha_n(x) \sim -x^{-2}/(2n+1)! \rightarrow 0, \quad x \rightarrow \infty. \quad (\text{A.8})$$

Again by Rolle's theorem, since $\frac{\partial}{\partial x} \alpha_n(x) \rightarrow 0$ and $\alpha_n(x) \rightarrow 0$ as $x \rightarrow \infty$, $Q_n(x)$ has a zero at some point $\xi_{n,n} < \chi_{n,n} < \infty$. For $x \geq 2$, the function $\sqrt{x(x-2)} \frac{\partial}{\partial x} \alpha_n(x)$ is well defined and it is zero for $x = 2$ and $x = \chi_{n,i}$, $i = 1, \dots, n$. With the same arguments as before, we then obtain that $\frac{\partial}{\partial x} [\sqrt{x(x-2)} \frac{\partial}{\partial x} \alpha_n(x)]$ is zero for $x = \xi_{n+1,i}$, $i = 1, \dots, n+1$, where $2 < \xi_{n+1,1} < \chi_{n,1} < \xi_{n+1,2} < \chi_{n,2} < \dots < \chi_{n,n} < \xi_{n+1,n+1} < \infty$. Then by Eq. (A.7), those zeros are also roots of, respectively, $\alpha_{n+1}(x)$ and $P_{n+1}(x)$. Since $P_{n+1}(x)$ is a polynomial of order $n+1$, it can have only $n+1$ roots, which were already found. Moreover, they are all real, distinct and strictly greater than 2, and the claim is proven. \square

Lemma A.2. *Suppose that $\Re(\mu_j) \neq 0$ for all $j = 1, \dots, q$. Then we have, for any $t \in (0, \infty)$ and $\omega \in \mathbb{R} \setminus \{0\}$,*

$$\begin{aligned} \lim_{\Delta \downarrow 0} \sigma \frac{\sqrt{\Delta}}{\sigma_\Delta} \frac{1 - \cos(\omega \lfloor t/\Delta \rfloor \Delta)}{\omega} \frac{\sin(\omega \Delta)}{1 - \cos(\omega \Delta)} \Re \left(\frac{\prod_{j=1}^p (1 - e^{\Delta(\lambda_j + i\omega)})}{\Theta_\Delta(e^{i\omega \Delta})} \frac{b(-i\omega)}{a(-i\omega)} \right) \\ = \frac{2 - 2\cos(\omega t)}{\omega^2} (1 + \Re(D(\omega))) \end{aligned}$$

and

$$\lim_{\Delta \downarrow 0} \sigma \frac{\sqrt{\Delta}}{\sigma_\Delta} \frac{1 - \cos(\omega \lfloor t/\Delta \rfloor \Delta)}{\omega} \Im \left(\frac{\prod_{j=1}^p (1 - e^{\Delta(\lambda_j + i\omega)})}{\Theta_\Delta(e^{i\omega \Delta})} \frac{b(-i\omega)}{a(-i\omega)} \right) = 0,$$

where $D(\omega) := -1 + \prod_{j \in J} (\mu_j - i\omega)/(-\mu_j - i\omega)$ and $J := \{j = 1, \dots, q : \Re(\mu_j) < 0\}$. Obviously, if $\Re(\mu_j) > 0$ for all $j = 1, \dots, q$, then $D(\omega) = 0$ for all $\omega \in \mathbb{R}$.

Proof. By Proposition 2.2, we have that $\eta(\xi_j) \in (0, 1)$ for sufficiently small Δ . Hence, for any $\omega \in \mathbb{R}$,

$$\begin{aligned} \frac{\prod_{j=1}^p (1 - e^{\Delta(\lambda_j + i\omega)})}{\Theta_\Delta(e^{i\omega \Delta})} \frac{b(-i\omega)}{a(-i\omega)} &= \frac{1}{\prod_{j=1}^{p-q-1} (1 + \eta(\xi_j) e^{i\omega \Delta})} \prod_{j=1}^p \frac{e^{\Delta(\lambda_j + i\omega)} - 1}{i\omega + \lambda_j} \prod_{j=1}^q \frac{\mu_j - i\omega}{1 - \zeta_j e^{i\omega \Delta}} \\ &= \Delta^{p-q} (1 + D(\omega)) \prod_{j=1}^{p-q-1} (1 + \eta(\xi_j))^{-1} \cdot (1 + o(1)) \quad \text{as } \Delta \downarrow 0. \end{aligned}$$

Moreover, using Eq. (2.10), we obtain

$$\sigma \frac{\sqrt{\Delta}}{\sigma_\Delta} = \frac{\sqrt{[2(p-q)-1]! \cdot \prod_{j=1}^{p-q-1} \eta(\xi_j)}}{\Delta^{p-q-1}} (1 + o(1)) \quad \text{as } \Delta \downarrow 0.$$

Since $\cos(\omega \lfloor t/\Delta \rfloor \Delta) \rightarrow \cos(\omega t)$ and $\Delta \sin(\omega \Delta)/(1 - \cos(\omega \Delta)) \rightarrow 2/\omega$ as $\Delta \downarrow 0$ for any $\omega \in \mathbb{R} \setminus \{0\}$, we can use the equality (cf. (Brockwell et al., 2012a, proof of Theorem 3.2))

$$\frac{\sqrt{[2(p-q)-1]! \cdot \prod_{j=1}^{p-q-1} \eta(\xi_j)}}{\prod_{j=1}^{p-q-1} (1 + \eta(\xi_j))} = \frac{\prod_{j=1}^{p-q-1} |1 + \eta(\xi_j)|}{\prod_{j=1}^{p-q-1} (1 + \eta(\xi_j))} \cdot (1 + o(1)) = 1 + o(1) \quad \text{as } \Delta \downarrow 0$$

to conclude the proof. \square

Lemma A.3. *Suppose that $t \in (0, \infty)$ and $\Re(\mu_j) \neq 0$ for all $j = 1, \dots, q$, and let the functions $h^{\Delta,1}(\cdot)$, $h^{\Delta,2}(\cdot)$ and $h^{\Delta,3}_{\lfloor t/\Delta \rfloor}(\cdot)$ be defined as in the proof of Theorem 3.2. Then there is a $C > 0$ such that, for any $\omega \in \mathbb{R}$ and any sufficiently small Δ ,*

$$\left| 2\Re \left(h^{\Delta,1} \cdot h^{\Delta,3}_{\lfloor t/\Delta \rfloor}(\omega) \cdot \overline{h^{\Delta,2} \cdot h^{\Delta,3}_{\lfloor t/\Delta \rfloor}(\omega)} \right) \right| \leq h(\omega),$$

where $h(\omega) := (7^{2p}/2^{2p+q} + 1)t^2 \mathbf{1}_{(-1,1)}(\omega) + C\omega^{-2} \mathbf{1}_{\mathbb{R} \setminus (-1,1)}(\omega)$. Moreover, h is integrable over the real line.

Proof. We obviously have

$$\left| 2\Re \left(h^{\Delta,1} \cdot h^{\Delta,3}_{\lfloor t/\Delta \rfloor}(\omega) \cdot \overline{h^{\Delta,2} \cdot h^{\Delta,3}_{\lfloor t/\Delta \rfloor}(\omega)} \right) \right| \leq \left| h^{\Delta,1} \cdot h^{\Delta,3}_{\lfloor t/\Delta \rfloor}(\omega) \right|^2 + \left| h^{\Delta,2} \cdot h^{\Delta,3}_{\lfloor t/\Delta \rfloor}(\omega) \right|^2 \quad (\text{A.9})$$

for any $\omega \in \mathbb{R}$ and any Δ . Let us first consider the second addend on the RHS of Eq. (A.9).

We obtain $|h^{\Delta,2} \cdot h^{\Delta,3}_{\lfloor t/\Delta \rfloor}(\omega)|^2 = 2(1 - \cos(\omega \lfloor t/\Delta \rfloor \Delta))/\omega^2$ and since $\lfloor t/\Delta \rfloor \Delta \leq t$ holds, we can bound, for any Δ , the latter function by t^2 on the interval $(-1, 1)$ and by $4/\omega^2$ on $\mathbb{R} \setminus (-1, 1)$.

As to the first addend on the RHS of Eq. (A.9), we calculate

$$\left| h^{\Delta,1} \cdot h^{\Delta,3}_{\lfloor t/\Delta \rfloor}(\omega) \right|^2 = \sigma^2 \frac{\Delta}{\sigma_\Delta^2} \frac{\prod_{j=1}^p |1 - e^{\Delta(\lambda_j + i\omega)}|^2}{|\Theta_\Delta(e^{i\omega\Delta})|^2} \frac{|b(-i\omega)|^2}{|a(-i\omega)|^2} \cdot \frac{1 - \cos(\omega \lfloor t/\Delta \rfloor \Delta)}{1 - \cos(\omega\Delta)}. \quad (\text{A.10})$$

Let now $|\omega| < 1$ and suppose that Δ is sufficiently small, i.e. the following inequalities will be true for any $|\omega| < 1$ whenever Δ is sufficiently small. Using $|1 - e^z| \leq 7/4|z|$ for $|z| < 1$ (see, e.g., (Abramowitz and Stegun, 1974, 4.2.38)) yields

$$\frac{\prod_{j=1}^p |1 - e^{\Delta(\lambda_j + i\omega)}|^2}{|a(-i\omega)|^2} \leq \left(\frac{7}{4} \Delta \right)^{2p}.$$

Then, $(1 - \cos(\omega\Delta))/(\omega\Delta)^2 \geq 1/4$ together with $4(1 - \cos(\omega \lfloor t/\Delta \rfloor \Delta))/\omega^2 \leq 2t^2$ (see above) implies

$$\frac{1 - \cos(\omega \lfloor t/\Delta \rfloor \Delta)}{1 - \cos(\omega\Delta)} \leq 2 \left(\frac{t}{\Delta} \right)^2.$$

As in the proof of Lemma A.2 we write $\Theta_\Delta(z) = \prod_{j=1}^{p-q-1} (1 + \eta(\xi_j)z) \cdot \prod_{j=1}^q (1 - \zeta_j z)$, where $\zeta_j = 1 - \text{sgn}(\Re(\mu_j)) \mu_j \Delta + o(\Delta)$ (see Brockwell et al. (2012a), Theorem 2.1). Since $\prod_{j=1}^q (|1 - \zeta_j e^{i\omega\Delta}|/\Delta)^2 \geq \prod_{j=1}^q 1/2 |\text{sgn}(\Re(\mu_j)) \mu_j - i\omega|^2$, we further deduce

$$\frac{|b(-i\omega)|^2}{\prod_{j=1}^q |1 - \zeta_j e^{i\omega\Delta}|^2} \leq \frac{2^q}{\Delta^{2q}}.$$

Again due to Eq. (2.10), we obtain

$$\sigma^2 \frac{\Delta}{\sigma_\Delta^2} \prod_{j=1}^{p-q-1} |1 + \eta(\xi_j) e^{i\omega\Delta}|^{-2} \leq \frac{2 \cdot [2(p-q) - 1]!}{\Delta^{2(p-q-1)}} \prod_{j=1}^{p-q-1} \frac{|\eta(\xi_j)|}{|1 + \eta(\xi_j) e^{i\omega\Delta}|^2}$$

and since $|\eta(\xi_j)| < 1$ for all j (see Proposition 2.2) we also have that $|1 + \eta(\xi_j) e^{i\omega\Delta}| \geq \frac{1}{2} |1 + \eta(\xi_j)|$ for all j , resulting in

$$\sigma^2 \frac{\Delta}{\sigma_\Delta^2} \prod_{j=1}^{p-q-1} |1 + \eta(\xi_j) e^{i\omega\Delta}|^{-2} \leq \frac{2^{2(p-q)-1}}{\Delta^{2(p-q-1)}} \cdot [2(p-q) - 1]! \prod_{j=1}^{p-q-1} \frac{|\eta(\xi_j)|}{|1 + \eta(\xi_j)|^2} = \frac{2^{2(p-q)-1}}{\Delta^{2(p-q-1)}}$$

where the latter equality follows from (Brockwell et al., 2012a, proof of Theorem 3.2). All together the RHS of Eq. (A.10) can be bounded for any $|\omega| < 1$ and any sufficiently small Δ by $(7/2)^{2p} 2^{-qt^2}$.

It remains to bound the RHS of Eq. (A.10) also for $|\omega| \geq 1$. Hence, for the rest of the proof let us suppose $|\omega| \geq 1$ and in addition we assume again that Δ is sufficiently small in order that all the following inequalities hold. We are going to show that

$$\sigma^2 \frac{\Delta}{\sigma_\Delta^2} \frac{\prod_{j=1}^p |1 - e^{\Delta(\lambda_j + i\omega)}|^2}{|\Theta_\Delta(e^{i\omega\Delta})|^2} \frac{|b(-i\omega)|^2}{|a(-i\omega)|^2} \frac{1 - \cos(\omega \lfloor t/\Delta \rfloor \Delta)}{1 - \cos(\omega\Delta)} \leq \frac{C}{\omega^2}$$

for some $C > 0$. Since $|\sigma^2 \Delta / \sigma_\Delta^2| \leq \text{const.} \cdot |\Delta^2 / \Delta^{2(p-q)}|$ (see (2.10)) and since $\prod_{j=1}^{p-q-1} |1 + \eta(\xi_j) e^{i\omega\Delta}|^{-2} \leq \prod_{j=1}^{p-q-1} (1 - |\eta(\xi_j)|)^{-2} \leq \text{const.}$ (cf. Proposition 2.2), it is sufficient to prove

$$\frac{(\omega\Delta)^2}{\Delta^{2(p-q)}} \frac{\prod_{j=1}^p |1 - e^{\Delta(\lambda_j + i\omega)}|^2}{\prod_{j=1}^q |1 - \zeta_j e^{i\omega\Delta}|^2} \frac{|b(-i\omega)|^2}{|a(-i\omega)|^2} \frac{1 - \cos(\omega \lfloor t/\Delta \rfloor \Delta)}{1 - \cos(\omega\Delta)} \leq C \quad (\text{A.11})$$

for some $C > 0$. The power transfer function satisfies $|b(-i\omega)|^2 / |a(-i\omega)|^2 \leq \text{const.} / (\omega^{2(p-q)} + 1)$ for any $\omega \in \mathbb{R}$. Thus, Eq. (A.11) will follow from

$$\frac{(\omega\Delta)^2}{(\omega\Delta)^{2(p-q)} + \Delta^{2(p-q)}} \frac{\prod_{j=1}^p |1 - e^{\Delta(\lambda_j + i\omega)}|^2}{\prod_{j=1}^q |1 - \zeta_j e^{i\omega\Delta}|^2} \frac{1 - \cos(\omega \lfloor t/\Delta \rfloor \Delta)}{1 - \cos(\omega\Delta)} \leq C. \quad (\text{A.12})$$

We even show that Eq. (A.12) is true for any $\omega \in \mathbb{R}$. However, using symmetry and periodicity arguments it is sufficient to prove Eq. (A.12) on the interval $[0, \frac{2\pi}{\Delta}]$. We split that interval into the following six subintervals

$$\begin{aligned} I_1 &:= \left[0, \min_{j=1, \dots, q} \frac{|\mu_j|}{2}\right], \quad I_2 := \left[\min_{j=1, \dots, q} \frac{|\mu_j|}{2}, \max_{j=1, \dots, q} 2|\mu_j|\right], \quad I_3 := \left[\max_{j=1, \dots, q} 2|\mu_j|, \frac{\pi}{\Delta}\right], \\ I_4 &:= \left[\frac{\pi}{\Delta}, \frac{2\pi}{\Delta} - \max_{j=1, \dots, q} 2|\mu_j|\right], \quad I_5 := \left[\frac{2\pi}{\Delta} - \max_{j=1, \dots, q} 2|\mu_j|, \frac{2\pi}{\Delta} - \min_{j=1, \dots, q} \frac{|\mu_j|}{2}\right] \text{ and} \\ I_6 &:= \left[\frac{2\pi}{\Delta} - \min_{j=1, \dots, q} \frac{|\mu_j|}{2}, \frac{2\pi}{\Delta}\right]. \end{aligned}$$

For any $\omega \in I_1 \cup I_6$, the fraction $\frac{1 - \cos(\omega \lfloor t/\Delta \rfloor \Delta)}{1 - \cos(\omega\Delta)}$ can be bounded by $\lfloor t/\Delta \rfloor^2$. In the other intervals we have the obvious bound $\frac{2}{1 - \cos(\omega\Delta)}$ for that term.

Now, for any $j = 1, \dots, p$, we have, as $\Delta \downarrow 0$,

$$\left|1 - e^{\Delta \lambda_j} \cdot e^{i\omega \Delta}\right|^2 \leq 2 \left|1 - e^{i\omega \Delta}\right|^2 + 4\Delta^2 |\lambda_j|^2 = 8 \sin^2 \left(\frac{\omega \Delta}{2}\right) + 4\Delta^2 |\lambda_j|^2 \leq 4\Delta^2 (\omega^2 + |\lambda_j|^2)$$

if $\omega \in I_1 \cup I_2 \cup I_3$, and $\left|1 - e^{\Delta \lambda_j} \cdot e^{i\omega \Delta}\right|^2 \leq 4\Delta^2 \left((2\pi/\Delta - \omega)^2 + |\lambda_j|^2\right)$ if $\omega \in I_4 \cup I_5 \cup I_6$.

The first fraction on the LHS of Eq. (A.12) satisfies

$$\frac{(\omega \Delta)^2}{(\omega \Delta)^{2(p-q)} + \Delta^{2(p-q)}} \leq \begin{cases} \min_{j=1, \dots, q} \frac{|\mu_j|}{2} \cdot \frac{\Delta^2}{\Delta^{2(p-q)}}, & \text{if } \omega \in I_1, \\ \frac{(\omega \Delta)^2}{(\omega \Delta)^{2(p-q)}}, & \text{if } \omega \in I_2 \cup I_3, \\ \frac{(2\pi)^2}{\pi^{2(p-q)}}, & \text{if } \omega \in I_4 \cup I_5 \cup I_6. \end{cases}$$

Then, for any $j = 1, \dots, q$ and $\omega \in I_1 \cup I_6$, we obtain

$$\begin{aligned} \left|1 - \zeta_j e^{i\omega \Delta}\right|^2 &= \left|1 - (1 - \operatorname{sgn}(\Re(\mu_j)) \mu_j \Delta + o(\Delta)) e^{i\omega \Delta}\right|^2 \geq \frac{1}{2} \Delta^2 |\operatorname{sgn}(\Re(\mu_j)) \mu_j - i\omega|^2 \\ &\geq \frac{1}{8} \Delta^2 |\mu_j|^2. \end{aligned}$$

If $\omega \in I_3$, then we have

$$\begin{aligned} \left|1 - \zeta_j e^{i\omega \Delta}\right|^2 &\geq (|1 - e^{i\omega \Delta}| - |\mu_j + o(1)| \Delta)^2 = \left(2 \sin \left(\frac{\omega \Delta}{2}\right) - |\mu_j + o(1)| \Delta\right)^2 \\ &\geq \Delta^2 \left(\frac{3}{5} \omega - |\mu_j + o(1)|\right)^2 \end{aligned}$$

and likewise, for $\omega \in I_4$, we deduce $\left|1 - \zeta_j e^{i\omega \Delta}\right|^2 \geq \Delta^2 \left(\frac{3}{5} \left(\frac{2\pi}{\Delta} - \omega\right) - |\mu_j + o(1)|\right)^2$. On I_2 we get for arbitrary $\varepsilon > 0$

$$\begin{aligned} \left|1 - \zeta_j e^{i\omega \Delta}\right|^2 &= 2(1 - \cos(\omega \Delta)) \cdot (1 - \Delta \operatorname{sgn}(\Re(\mu_j)) \Re(\mu_j) + o(\Delta)) \\ &\quad + 2 \sin(\omega \Delta) \cdot (-\Delta \operatorname{sgn}(\Re(\mu_j)) \Im(\mu_j) + o(\Delta)) + \Delta^2 |\mu_j|^2 + o(\Delta^2) \\ &\geq (\omega \Delta)^2 \cdot (1 - \varepsilon) - 2(\omega \Delta) \cdot \Delta |\Im(\mu_j)| \cdot (1 + \varepsilon) + \Delta^2 (|\mu_j|^2 + o(1)) =: f_\varepsilon^\Delta(\omega \Delta). \end{aligned}$$

Since $f_\varepsilon^\Delta(\omega)/\omega^2 \rightarrow 1 - \varepsilon$ ($\omega \rightarrow \infty$) and $f_\varepsilon^\Delta(\omega)/\omega^2 \rightarrow \infty$ ($\omega \rightarrow 0$), a (global) minimum of $f_\varepsilon^\Delta(\omega)/\omega^2$ on $(0, \infty)$ could be achieved in any ω^* with $\left(\frac{d}{d\omega} \frac{f_\varepsilon^\Delta(\omega)}{\omega^2}\right)(\omega^*) = 0$. The only such value is $\omega^* = \frac{\Delta(|\mu_j|^2 + o(1))}{(1+\varepsilon)|\Im(\mu_j)|}$. Since

$$\frac{f_\varepsilon^\Delta(\omega^*)}{(\omega^*)^2} = 1 - \varepsilon - (1 + \varepsilon)^2 \frac{|\Im(\mu_j)|^2}{|\mu_j|^2 + o(1)} \geq (1 + \varepsilon) \frac{\Re(\mu_j)^2}{|\mu_j|^2} - 3\varepsilon - \varepsilon^2 \geq \frac{1}{2} \frac{\Re(\mu_j)^2}{|\mu_j|^2};$$

if we choose $\varepsilon = \frac{1}{6} \frac{\Re(\mu_j)^2}{|\mu_j|^2}$, we obtain $\frac{f_\varepsilon^\Delta(\omega)}{\omega^2} \geq \frac{1}{2} \frac{\Re(\mu_j)^2}{|\mu_j|^2}$ for any $\omega \in (0, \infty)$. Hence,

$$\left|1 - \zeta_j e^{i\omega \Delta}\right|^2 \geq f_\varepsilon^\Delta(\omega \Delta) \geq \frac{1}{2} \frac{\Re(\mu_j)^2}{|\mu_j|^2} (\omega \Delta)^2 \quad \text{for all } \omega \in I_2.$$

Using periodic properties of the sine and cosine terms, we likewise get

$$\left|1 - \zeta_j e^{i\omega \Delta}\right|^2 \geq \frac{1}{2} \frac{\Re(\mu_j)^2}{|\mu_j|^2} \Delta^2 \left(\frac{2\pi}{\Delta} - \omega\right)^2 \quad \text{for any } \omega \in I_5.$$

Putting all together, we can bound the LHS of Eq. (A.12) in I_1 by

$$\begin{aligned} \min_{j=1,\dots,q} \frac{|\mu_j|}{2} \cdot \frac{(\lfloor t/\Delta \rfloor \Delta)^2}{\Delta^{2(p-q)}} \frac{4^p \Delta^{2p} \cdot \prod_{j=1}^p (\min_{k=1,\dots,q} |\mu_k|^2/4 + |\lambda_j|^2)}{8^{-q} \Delta^{2q} \prod_{j=1}^q |\mu_j|^2} \\ \leq \min_{j=1,\dots,q} \frac{|\mu_j|}{2} \cdot t^2 \cdot \frac{4^{p+q} \cdot \prod_{j=1}^p (\min_{k=1,\dots,q} |\mu_k|^2/4 + |\lambda_j|^2)}{\prod_{j=1}^q \frac{1}{2} |\mu_j|^2} = C, \end{aligned}$$

in I_2 by

$$\begin{aligned} \frac{2(\omega\Delta)^2}{1 - \cos(\omega\Delta)} \frac{4^p \Delta^{2p} \cdot \prod_{j=1}^p (4 \max_{k=1,\dots,q} |\mu_k|^2 + |\lambda_j|^2)}{(\omega\Delta)^{2p} \cdot \prod_{j=1}^q \frac{1}{2} \frac{\Re(\mu_j)^2}{|\mu_j|^2}} \\ \leq \frac{5 \cdot 4^{2p} \cdot \prod_{j=1}^p (4 \max_{k=1,\dots,q} |\mu_k|^2 + |\lambda_j|^2)}{\min_{j=1,\dots,q} |\mu_j|^{2p} \cdot \prod_{j=1}^q \frac{1}{2} \frac{\Re(\mu_j)^2}{|\mu_j|^2}} = C, \end{aligned}$$

in I_3 by

$$\begin{aligned} \frac{2(\omega\Delta)^2}{1 - \cos(\omega\Delta)} \frac{4^p (\omega\Delta)^{2p} \cdot \prod_{j=1}^p \left(1 + \frac{|\lambda_j|^2}{4 \max_{k=1,\dots,q} |\mu_k|^2}\right)}{(\omega\Delta)^{2(p-q)} \cdot \left(\frac{1}{20} \omega\Delta\right)^{2q}} \\ \leq \pi^2 4^p 20^{2q} \prod_{j=1}^p \left(1 + \frac{|\lambda_j|^2}{4 \max_{k=1,\dots,q} |\mu_k|^2}\right) = C, \end{aligned}$$

in I_4 by

$$\begin{aligned} \frac{(2\pi)^2}{\pi^{2(p-q)}} \frac{2}{1 - \cos(\omega\Delta)} \frac{4^p (2\pi - \omega\Delta)^{2p} \cdot \prod_{j=1}^p \left(1 + \frac{|\lambda_j|^2}{4 \max_{k=1,\dots,q} |\mu_k|^2}\right)}{20^{-2q} (2\pi - \omega\Delta)^{2q}} \\ \leq 4^{p+1} 20^{2q} \prod_{j=1}^p \left(1 + \frac{|\lambda_j|^2}{4 \max_{k=1,\dots,q} |\mu_k|^2}\right) \frac{2 \cdot (2\pi - \omega\Delta)^2}{1 - \cos(2\pi - \omega\Delta)} \\ \leq \pi^2 4^{p+1} 20^{2q} \prod_{j=1}^p \left(1 + \frac{|\lambda_j|^2}{4 \max_{k=1,\dots,q} |\mu_k|^2}\right) = C, \end{aligned}$$

in I_5 by

$$\begin{aligned} \frac{(2\pi)^2}{\pi^{2(p-q)}} \frac{2}{1 - \cos(\omega\Delta)} \frac{4^p \Delta^{2p} \cdot \prod_{j=1}^p (4 \max_{k=1,\dots,q} |\mu_k|^2 + |\lambda_j|^2)}{\Delta^{2q} \prod_{j=1}^q \frac{1}{8} \min_{k=1,\dots,q} |\mu_k|^2 (\Re(\mu_j)/|\mu_j|)^2} \\ \leq \frac{(2\pi)^2}{\pi^{2(p-q)}} \frac{4^p \cdot \prod_{j=1}^p (4 \max_{k=1,\dots,q} |\mu_k|^2 + |\lambda_j|^2)}{\prod_{j=1}^q \frac{1}{8} \min_{k=1,\dots,q} |\mu_k|^2 (\Re(\mu_j)/|\mu_j|)^2} \frac{2\Delta^2}{1 - \cos(2\pi - \omega\Delta)} \\ \leq \frac{(2\pi)^2}{\pi^{2(p-q)}} \frac{4^p \cdot \prod_{j=1}^p (4 \max_{k=1,\dots,q} |\mu_k|^2 + |\lambda_j|^2)}{\prod_{j=1}^q \frac{1}{8} \min_{k=1,\dots,q} |\mu_k|^2 (\Re(\mu_j)/|\mu_j|)^2} \frac{5 \cdot 4}{\min_{j=1,\dots,q} |\mu_j|^2} = C, \end{aligned}$$

and, finally, in I_6 by

$$\begin{aligned} \frac{(2\pi \lfloor t/\Delta \rfloor)^2}{\pi^{2(p-q)}} \frac{4^p \Delta^{2p} \cdot \prod_{j=1}^p (\min_{k=1,\dots,q} |\mu_k|^2/4 + |\lambda_j|^2)}{8^{-q} \Delta^{2q} \prod_{j=1}^q |\mu_j|^2} \\ \leq \frac{(2\pi t)^2}{\pi^{2(p-q)}} \frac{4^{p+q} \cdot \prod_{j=1}^p (\min_{k=1,\dots,q} |\mu_k|^2/4 + |\lambda_j|^2)}{\prod_{j=1}^q \frac{1}{2} |\mu_j|^2} = C. \end{aligned}$$

This shows Eq. (A.12) and thus concludes the proof. \square

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